Three loop anomalous dimensions of higher moments of the non-singlet twist-2 Wilson and transversity operators in the $\overline{\text{MS}}$ and RI$'$ schemes

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ABSTRACT: We compute the anomalous dimension of the third and fourth moments of the flavour non-singlet twist-2 Wilson and transversity operators at three loops in both the $\overline{\text{MS}}$ and RI$'$ schemes. To assist with the extraction of estimates of matrix elements computed using lattice regularization, the finite parts of the Green's function where the operator is inserted in a quark 2-point function are also provided at three loops in both schemes.

KEYWORDS: NLO Computations, Renormalization Group, Deep Inelastic Scattering.

Contents

1. Introduction

In calculations relating to deep inelastic scattering the operator product expansion plays an important role in allowing one to evaluate the underlying current correlators. In essence there are two parts to the formalism. The first is the basis of gauge invariant operators into which the current correlators are decomposed and the other is the process dependent Wilson coefficients. For high energy experiments the dominant set of operators are those of leading twist defined as the difference between dimension and spin. The Wilson coefficients are determined from the specific process of interest and are computed in perturbation theory. To understand the physics at various energies one requires the solution of the underlying renormalization group equation at some particular loop approximation and necessary for this are the anomalous dimensions of the operators of the basis. With the increased precision now demanded by experiments the goal in recent years has been to ascertain the anomalous dimensions at three loops as an analytic function of the moment, n, of the operators. This has been achieved by the magnificent computation of [1−4] for the twist-2 flavour non-singlet and singlet Wilson operators in the $\overline{\text{MS}}$ scheme as well as the Wilson coefficients to the same order which extended the lower loop results of [5−7]. Hence the *full* two loop renormalization group evolution has been determined. However, one feature which cannot be deduced from perturbative techniques is the underlying matrix element which is non-perturbative in nature and has to be deduced by use of the lattice where various moments of the matrix element have been determined for low moments.

See, for example, the ongoing work of the QCDSF collaboration, [8 – 13], and others, [14 – 17]. However, in making measurements of matrix elements, which ultimately will provide accurate predictions for experiments, there are several issues.

First, to reduce computation time and consequently cost, the matrix elements are determined on the lattice in a renormalization scheme geared for this problem which is known as the regularization invariant (RI) scheme and its variation called the modified regularization invariant (RI′) scheme, [18]. Both are mass dependent renormalization schemes. However, when the results are extracted in this scheme they need to be converted to the reference $\overline{\text{MS}}$ scheme which is a mass independent scheme. An early example of the application of this approach is given in [19, 20]. Second, to ensure that the results are credible when evolved to large energy they must match the perturbative result for the matrix element in the *same* renormalization scheme. There are several ways of achieving this. One is to use a non-perturbative approach such as the Schrödinger functional method, [21]. The other is to compute the matrix element to as many orders as possible in conventional perturbation theory and then match the lattice results to the explicit perturbative results in the same renormalization scheme. Previously that has been the approach of Chetyrkin and Rétey, [20], and [22, 23]. Specifically, various quark currents have been considered including the tensor current as well as the second moment of the flavour non-singlet Wilson and transversity operators. The latter was originally introduced in [24 – 26] and relates to the probability of finding a quark in a transversely polarized nucleon polarized parallel to the nucleon versus that of the nucleon in the antiparallel polarization. The results of [20, 22, 23] have been important in the matrix element lattice computations of [11 – 17]. Necessary for the three loop perturbative calculations of the Green's functions with the operator inserted has been the full three loop renormalization of QCD in the RI′ scheme in an arbitrary linear covariant gauge, [20, 22]. Given that there has been an advance in computing technology in recent years, it transpires that it is now feasible to measure higher moments of the underlying matrix elements on the lattice since essentially there has been a significant improvement in numerically isolating a clear signal. Therefore to assist the full matching procedure in the ultraviolet region to produce *accurate* estimates, it is necessary to extend the approach of [22, 23] to higher moments of these two classes of operators. This is the purpose of this article where we will consider the third and fourth moments of the flavour non-singlet Wilson and transversity operators in an arbitrary linear covariant gauge. There will be two parts to this venture. The first is the determination of the anomalous dimensions of the operators at three loops in $\overline{\text{MS}}$ and RI′. Whilst the second is to produce to the same loop order the value of the Green's function involving the operator itself inserted in a quark 2-point function in both schemes. Although it is worth noting that for the lattice, only the Landau gauge results are relevant since that is the gauge in which lattice measurements are made. The full arbitrary gauge calculation, being more complete, is actually important for internal checks on the loop computations. Whilst the even moments are appropriate for deep inelastic scattering experiments, the odd moments are accessible on the lattice and can serve the purpose of a testing ground for technical issues for higher moment lattice matching. Whilst the Wilson operator anomalous dimensions are known already at three loops in $\overline{\text{MS}}$, [1, 2], we note that what is required are the values

of the specific Green's function with the operator inserted which has not been determined previously. In also considering the transversity operator, we will deduce *new* anomalous dimensions at three loops in both $\overline{\text{MS}}$ and RI′ beyond the earlier two loop calculation of [27 – 31]. Moreover, since we will be using symbolic manipulation and computer algebra tools and given that the Wilson and transversity operators are both formally similar, the actual computations are efficiently performed through the same computer programmes.

The paper is organised as follows. In section 2 we review the basics of the RI′ scheme and introduce the properties of the operators we will consider. The full three loop anomalous dimensions for these operators are given in both schemes in section 3, whilst the same information for the underlying Green's functions are provided in section 4 including the restriction to the colour group SU(3) and the Landau gauge. Section 5 records the explicit functions which convert all the three loop anomalous dimensions between the $\overline{\text{MS}}$ and RI′ schemes with concluding remarks given in section 6. Several appendices summarize the projection of the Green's function onto the basis of independent Lorentz tensors sharing the same symmetry properties as the corresponding operator, as well as the construction of the full operator with these symmetries.

2. RI′ scheme

In this section we discuss the definition of the RI′ scheme and properties of the operators we are interested in. First, we recall that the standard renormalization scheme is the MS scheme, [32], where the poles with respect to the regulator are absorbed into the renormalization constants. Its widely used extension, $\overline{\text{MS}}$, which is also a mass independent scheme additionally absorbs the finite part $\ln(4\pi e^{-\gamma})$ where γ is the Euler-Mascheroni constant, [33]. By contrast the regularization invariant schemes, [18], are mass dependent schemes which from the point of view of the Lagrangian is the scheme where the quark 2-point function not only is rendered finite but also chosen to be unity through the RI definition

$$\lim_{\epsilon \to 0} \left[\frac{1}{4d} \text{tr} \left(Z_\psi^{\text{RI}} \gamma^\mu \frac{\partial}{\partial p^\mu} \Sigma_\psi(p) \right) \right] \Bigg|_{p^2 = \mu^2} = 1 \ . \tag{2.1}$$

where $\Sigma_\psi(p)$ is the quark 2-point function with external momentum p, $Z_\psi^{\mathcal{R}}$ is the quark wave function renormalization constant in the $\mathcal{R}$ renormalization scheme and μ is the scale introduced to ensure that the coupling constant remains dimensionless in d dimensions when using dimensional regularization with $d = 4 - 2\epsilon$. However, in practice since taking a derivative is (financially) costly on the lattice, one takes a variation of this definition (2.1), to define the RI′ scheme which does not involve differentiation, [18], which is

$$\lim_{\epsilon \to 0} \left[Z_\psi^{\text{RI}'} \Sigma_\psi(p) \right] \Bigg|_{p^2 = \mu^2} = \not{p} \ . \tag{2.2}$$

Although this is primarily the key to defining the scheme on the lattice as well as the continuum, the full three loop renormalization of QCD has been performed in an arbitrary linear covariant gauge and colour group in [22]. Additionally part of the four loop renormalization has been performed for SU(N_c) in [20]. However, in [22] the other field 2-point

functions were defined in an analogous way to (2.2). Further, by contrast the 3-point functions were renormalized according to the usual $\overline{\text{MS}}$ condition. So that those Green's functions were not constrained to be unity. Consequently, the relationship between the variables in both schemes were established to three loops and specifically are, [22],

$$a_{\text{RI}'} \;=\; a_{\overline{\text{MS}}} \;+\; O\left(a_{\overline{\text{MS}}}^5\right) \tag{2.3}$$

where $a = g^2/(16\pi^2)$ in terms of the coupling constant g in the definition of the covariant derivative D_μ, and for the arbitrary linear covariant gauge parameter α,

$$
\begin{aligned}
\alpha_{\text{RI}'} = \Big[1 &+ \Big(\left(-9\alpha_{\overline{\text{MS}}}^2 - 18\alpha_{\overline{\text{MS}}} - 97 \right) C_A + 80 T_F N_f \Big) \frac{a_{\overline{\text{MS}}}}{36} \\
&+ \Big(\left(18\alpha_{\overline{\text{MS}}}^4 - 18\alpha_{\overline{\text{MS}}}^3 + 190\alpha_{\overline{\text{MS}}}^2 - 576\zeta(3)\alpha_{\overline{\text{MS}}} + 463\alpha_{\overline{\text{MS}}} + 864\zeta(3) - 7143 \right) C_A^2 \\
&\quad + \left(- 320\alpha_{\overline{\text{MS}}}^2 - 320\alpha_{\overline{\text{MS}}} + 2304\zeta(3) + 4248 \right) C_A T_F N_f \\
&\quad + \left(- 4608\zeta(3) + 5280 \right) C_F T_F N_f \Big) \frac{a_{\overline{\text{MS}}}^2}{288} \\
&+ \Big(\Big(- 486\alpha_{\overline{\text{MS}}}^6 + 1944\alpha_{\overline{\text{MS}}}^5 + 4212\zeta(3)\alpha_{\overline{\text{MS}}}^4 - 5670\zeta(5)\alpha_{\overline{\text{MS}}}^4 - 18792\alpha_{\overline{\text{MS}}}^4 \\
&\quad + 48276\zeta(3)\alpha_{\overline{\text{MS}}}^3 - 6480\zeta(5)\alpha_{\overline{\text{MS}}}^3 - 75951\alpha_{\overline{\text{MS}}}^3 - 52164\zeta(3)\alpha_{\overline{\text{MS}}}^2 \\
&\quad + 2916\zeta(4)\alpha_{\overline{\text{MS}}}^2 + 124740\zeta(5)\alpha_{\overline{\text{MS}}}^2 + 92505\alpha_{\overline{\text{MS}}}^2 - 1303668\zeta(3)\alpha_{\overline{\text{MS}}} \\
&\quad + 11664\zeta(4)\alpha_{\overline{\text{MS}}} + 447120\zeta(5)\alpha_{\overline{\text{MS}}} + 354807\alpha_{\overline{\text{MS}}} + 2007504\zeta(3) \\
&\quad + 8748\zeta(4) + 1138050\zeta(5) - 10221367 \Big) C_A^3 \\
&\quad + \Big(12960\alpha_{\overline{\text{MS}}}^4 - 8640\alpha_{\overline{\text{MS}}}^3 - 129600\zeta(3)\alpha_{\overline{\text{MS}}}^2 - 147288\alpha_{\overline{\text{MS}}}^2 \\
&\quad\quad + 698112\zeta(3)\alpha_{\overline{\text{MS}}} - 312336\alpha_{\overline{\text{MS}}} + 1505088\zeta(3) - 279936\zeta(4) \\
&\quad\quad - 1658880\zeta(5) + 9236488 \Big) C_A^2 T_F N_f \\
&\quad + \Big(248832\zeta(3)\alpha_{\overline{\text{MS}}}^2 - 285120\alpha_{\overline{\text{MS}}}^2 + 248832\zeta(3)\alpha_{\overline{\text{MS}}} - 285120\alpha_{\overline{\text{MS}}} \\
&\quad\quad - 5156352\zeta(3) + 373248\zeta(4) - 2488320\zeta(5) + 9293664 \Big) C_A C_F T_F N_f \\
&\quad + \Big(- 38400\alpha_{\overline{\text{MS}}}^2 - 221184\zeta(3)\alpha_{\overline{\text{MS}}} + 55296\alpha_{\overline{\text{MS}}} \\
&\quad\quad - 884736\zeta(3) - 1343872 \Big) C_A T_F^2 N_f^2 \\
&\quad + \left(- 3068928\zeta(3) + 4976640\zeta(5) - 988416 \right) C_F^2 T_F N_f \\
&\quad + \left(2101248\zeta(3) - 2842368 \right) C_F T_F^2 N_f^2 \Big) \frac{a_{\overline{\text{MS}}}^3}{31104} \Big] \alpha_{\overline{\text{MS}}} \;+\; O\left(a_{\overline{\text{MS}}}^4\right)
\end{aligned} \tag{2.4}
$$

where T_F, C_F and C_A are the usual group theory factors defined by

$$\text{Tr}\left(T^a T^b \right) \;=\; T_F \delta^{ab} \;, \quad T^a T^a \;=\; C_F I \;, \quad f^{acd} f^{bcd} \;=\; C_A \delta^{ab} \tag{2.5}$$

for colour group generators T^a, $\zeta(n)$ is the Riemann zeta function and the scheme of the variable is indicated as a subscript. Clearly only in the Landau gauge, where $\alpha = 0$, are the variables equivalent. Although ultimately we are interested in the Landau gauge for the lattice matching, we have chosen to compute in an arbitrary linear covariant gauge as the

extra α dependence, evident in expressions such as (2.4), will provide a central checking tool in the loop computations. For instance, in a mass independent renormalization scheme the anomalous dimension of a gauge invariant operator is independent of the gauge parameter. Though this is not the case for mass dependent schemes such as RI$'$ which will be apparent in our explicit results.

More specifically the non-singlet operators we will focus on are

$$\mathcal{O}_W^{\nu_1\dots\nu_n} = \mathcal{S}\bar{\psi}\gamma^{\nu_1}D^{\nu_2}\dots D^{\nu_n}\psi$$
$$\mathcal{O}_T^{\mu\nu_1\dots\nu_n} = \mathcal{S}\bar{\psi}\sigma^{\mu\nu_1}D^{\nu_2}\dots D^{\nu_n}\psi \tag{2.6}$$

with $n = 3$ and 4 where $\sigma^{\mu\nu} = \frac{1}{2}[\gamma^\mu,\gamma^\nu]$ and is antisymmetric in its Lorentz indices. The operator $\mathcal{S}$ in both sets denotes the symmetrization of the Lorentz indices $\{\nu_1,\dots,\nu_n\}$ and the selection of the traceless part according to slightly different criteria for both cases. For the Wilson operators this is

$$\eta_{\nu_i\nu_j}\mathcal{O}_W^{\nu_1\dots\nu_i\dots\nu_j\dots\nu_n} = 0 . \tag{2.7}$$

Whilst for the transversity operators, [29],

$$\eta_{\mu\nu_i}\mathcal{O}_T^{\mu\nu_1\dots\nu_i\dots\nu_n} = 0 \quad (i \geq 2)$$
$$\eta_{\nu_i\nu_j}\mathcal{O}_T^{\mu\nu_1\dots\nu_i\dots\nu_j\dots\nu_n} = 0 . \tag{2.8}$$

Therefore, the transversity operator has formally one less traceless condition compared to the Wilson operators.

For the renormalization of an operator in a quark 2-point function, which will be either $\langle\psi(-p)\mathcal{O}_W^{\mu_1\dots\mu_n}(0)\bar{\psi}(p)\rangle$ or $\langle\psi(-p)\mathcal{O}_T^{\mu_1\dots\nu_n}(0)\bar{\psi}(p)\rangle$, the RI$'$ scheme definition is similar to (2.2), [18–20, 22, 23]. However, as the operators we will consider will carry Lorentz indices this 2-point function will decompose into several invariant amplitudes which may or may not have a tree (T) or Born term. For those amplitudes which have a tree term, the RI$'$ scheme definition is, [23],

$$\lim_{\epsilon\to 0}\left[Z_\psi^{\mathrm{RI}'}Z_\mathcal{O}^{\mathrm{RI}'}\Sigma_\mathcal{O}^{(T)}(p)\right]\Big|_{p^2 = \mu^2} = \mathcal{T} \tag{2.9}$$

where $\Sigma_\mathcal{O}^{(T)}(p)$ is the tree part of $\langle\psi(-p)\mathcal{O}(0)\bar{\psi}(p)\rangle$ and $\mathcal{T}$ is the value of the tree term amplitude which may not necessarily be unity. In other words there is no a dependence in $\mathcal{T}$. The explicit details of our choice of how to construct the Green's functions in terms of a basis of Lorentz tensors satisfying the same symmetry properties as the original operator are given in appendix A. This summarizes the procedure we will use to extract the renormalization constants of the operators which will then be encoded in the associated anomalous dimensions through the respective definitions

$$\gamma_\mathcal{O}^{\overline{\mathrm{MS}}}(a) = -\beta(a)\frac{\partial\ln Z_\mathcal{O}^{\overline{\mathrm{MS}}}}{\partial a} - \alpha\gamma_\alpha^{\overline{\mathrm{MS}}}(a)\frac{\partial\ln Z_\mathcal{O}^{\overline{\mathrm{MS}}}}{\partial\alpha} \tag{2.10}$$

and

$$\gamma_\mathcal{O}^{\mathrm{RI}'}(a) = -\beta(a)\frac{\partial\ln Z_\mathcal{O}^{\mathrm{RI}'}}{\partial a} - \alpha\gamma_\alpha^{\mathrm{RI}'}(a)\frac{\partial\ln Z_\mathcal{O}^{\mathrm{RI}'}}{\partial\alpha} \tag{2.11}$$

where $\gamma_\alpha^{\mathrm{RI}'}(a)$ is given in [22].

3. Anomalous dimensions

Having described in detail the method of renormalizing in the RI′ scheme, we now record the explicit three loop results for the anomalous dimensions in this section. In constructing our results we made extensive use of the MINCER package, [34, 35], written in the symbolic manipulation language FORM, [36]. The MINCER algorithm, [34], determines the divergent and finite parts of massless 2-point functions using dimensional regularization in d-dimensions. Therefore, it is ideal for the current problem since the Green's functions we are interested in are massless quark 2-point functions with the appropriate operator inserted at zero momentum. This is the momentum configuration which is measured on the lattice. Moreover, since we are concerned with operators which are symmetrized in their Lorentz indices and satisfy various tracelessness conditions in addition to being flavour non-singlet operators, there is no possibility of mixing into other operators. This is an important observation since ordinarily nullifying the momentum flow through the operator could lead to the inability to correctly determine the projection into the full basis of operators. (For a clear exposition on the deeper perils of operator mixing see, for example, [37].) The fact that each of the operators is multiplicatively renormalizable avoids this potential technicality. For our three loop computation we generated the Feynman diagrams with the QGRAF package, [38]. Specifically there are 3 one loop, 37 two loop and 684 three loop diagrams to be calculated. Though for the operators with no three gluon and two quark leg operator insertions the latter total is reduced by 14. Finally, the electronic QGRAF output is converted into FORM input notation and the FORM version of the MINCER algorithm, [35], is applied to the 724 Feynman diagrams. The actual Feynman rules for each operator were generated automatically in FORM. First we constructed the object with the same symmetry and traceless conditions as the operators we are interested in. The explicit details for each operator are given in appendix B. Then we applied an algorithm which systematically substitutes for the covariant derivatives and functionally differentiates with respect to the various fields present to produce electronic forms of the 2, 3, 4 and 5-point operator vertex insertions.

Now we provide our results in $\overline{\text{MS}}$. For completeness we give those for the two Wilson operators and note that we found exact agreement with the results first deduced in [1, 5, 6, 39, 40]. These are

$$
\begin{aligned}
\gamma^{\overline{\text{MS}}}_{\bar{\psi}\gamma^\mu D^\nu D^\sigma \psi}(a) = {} & \frac{25}{6}C_F a \;+\; C_F \left[\frac{535}{27}C_A - \frac{2035}{432}C_F - \frac{415}{54}T_F N_f \right] a^2 \\
& + C_F \left[\left(\frac{55}{3}\zeta(3) + \frac{889433}{7776} \right) C_A^2 - \left(55\zeta(3) + \frac{311213}{15552} \right) C_A C_F \right. \\
& \quad - \left(\frac{200}{3}\zeta(3) + \frac{62249}{1944} \right) C_A T_F N_f + \left(\frac{110}{3}\zeta(3) - \frac{244505}{15552} \right) C_F^2 \\
& \quad \left. + \left(\frac{200}{3}\zeta(3) - \frac{203627}{3888} \right) C_F T_F N_f - \frac{2569}{486}T_F^2 N_f^2 \right] a^3 \\
& + O(a^4)
\end{aligned}
\tag{3.1}
$$

and

$$\gamma^{\overline{\text{MS}}}_{\bar{\psi}\gamma^\mu D^\nu D^\sigma D^\rho \psi}(a) = \frac{157}{30}C_F a \ + \ [1292560C_A - 287303C_F - 530840T_F N_f] \frac{C_F a^2}{54000}$$
$$+ \ \left[(932472000\zeta(3) + 6803318650)\, C_A^2\right.$$
$$- \ (2797416000\zeta(3) + 1335140785)\, C_A C_F$$
$$- \ (4069440000\zeta(3) + 1760516200)\, C_A T_F N_f$$
$$+ \ (1864944000\zeta(3) - 714245693)\, C_F^2$$
$$+ \ (4069440000\zeta(3) - 3304751260)\, C_F T_F N_f$$
$$- \ \left. 307421600 T_F^2 N_f^2 \right] \frac{C_F a^3}{48600000} \ + \ O(a^4) \qquad (3.2)$$

where we note that throughout the article when the operator appears explicitly as a subscript on an object, it is regarded as a label and the free indices do not endow the object with tensor properties. Likewise when we indicate the renormalization scheme on a quantity which is evaluated in perturbation theory, that means that the variables in which it is expressed, such as a and α, are regarded as the variables in the *same* scheme. The relationship between the variables in either scheme is given in (2.3) and (2.4). For the two transversity operators the $\overline{\text{MS}}$ anomalous dimensions have not been given previously and we find that

$$\gamma^{\overline{\text{MS}}}_{\bar{\psi}\sigma^{\mu\nu} D^\sigma D^\rho \psi}(a) = \frac{13}{3}C_F a \ + \ [1195C_A - 311C_F - 452T_F N_f] \frac{C_F a^2}{54}$$
$$+ \ \left[(10368\zeta(3) + 126557)\, C_A^2 - (31104\zeta(3) + 30197)\, C_A C_F\right.$$
$$- \ (67392\zeta(3) + 38900)\, C_A T_F N_f + (67392\zeta(3) - 50552)\, C_F T_F N_f$$
$$+ \ \left. (20736\zeta(3) - 17434)\, C_F^2 - 4816 T_F^2 N_f^2 \right] \frac{C_F a^3}{972} \ + \ O(a^4) \qquad (3.3)$$

and

$$\gamma^{\overline{\text{MS}}}_{\bar{\psi}\sigma^{\mu\nu} D^\sigma D^\rho D^\lambda \psi}(a) = \frac{16}{3}C_F a \ + \ [1357C_A - 296C_F - 554T_F N_f] \frac{C_F a^2}{54}$$
$$+ \ \left[(272160\zeta(3) + 2893009)\, C_A^2 \ - \ (816480\zeta(3) + 662155)\, C_A C_F\right.$$
$$- \ (1658880\zeta(3) + 798892)\, C_A T_F N_f$$
$$+ \ (544320\zeta(3) - 235100)\, C_F^2$$
$$+ \ (1658880\zeta(3) - 1328860)\, C_F T_F N_f$$
$$- \ \left. 117776 T_F^2 N_f^2 \right] \frac{C_F a^3}{19440} \ + \ O(a^4) \,. \qquad (3.4)$$

There are several checks on these two expressions. First, as we have computed them in an arbitrary covariant gauge their final form must be independent of the gauge parameter in a mass independent renormalization scheme, which is apparent in (3.3) and (3.4), [32, 41]. Second, part of each of the three loop terms has in fact already been determined by the large N_f expansion in [23]. There the leading order critical exponent corresponding to the anomalous dimension evaluated at the non-trivial d-dimensional fixed point of the QCD β-function was determined in d-dimensions using a method that was originally developed to

study the perturbative structure of scalar field theories, [42, 43]. This critical exponent, [23], encodes all orders information on the corresponding renormalization group function at $O(1/N_f)$. Therefore, if we formally write the leading $O(1/N_f)$ part of the arbitrary n transversity operator anomalous dimension as, [23],

$$\gamma^{(n)\overline{\text{MS}}}_{\bar{\psi}\sigma^{\mu\nu_1}D^{\nu_2}\dots D^{\nu_n}\psi}(a) = C_F \left[b_1 a + (b_{21}T_F N_f + b_{20}) a^2 + \sum_{r=3}^{\infty}\sum_{j=0}^{r-1} b_{rj} T_F^j N_f^j a^r \right] \quad (3.5)$$

then the leading order coefficient of the N_f polynomial at three loops is given by

$$b_{32} = \frac{4}{27}\left[48 S_3(n) - 80 S_2(n) - 16 S_1(n) + \frac{3(17n^2 + 17n - 8)}{n(n+1)} \right] \quad (3.6)$$

where $S_l(n) = \sum_{i=1}^{n} 1/i^l$. Evaluating this for $n = 3$ and $n = 4$ reproduces the corresponding coefficients in (3.3) and (3.4) respectively. As a final check we note that we have used the method of [44] to perform the automatic renormalization of Green's functions at high loop order. This entails computing the Green's functions as a function of the bare coupling constant and bare gauge parameter. The counterterms are then introduced at the end of the computation by rescaling by the known renormalization constants. Therefore, the remaining divergence is absorbed by the renormalization constant associated with the Green's function. Moreover, given the way it has been deduced, the non-simple poles in ϵ are constrained to satisfy a specific form depending on the lower order simple poles due to the underlying renormalization group equation. This is a non-trivial checking criterion, especially in the presence of parameters such as the gauge parameter and group Casimirs, and it is reassuring to record that all the renormalization constants determined for the above anomalous dimensions precisely satisfied this criterion. Implicit in this final check is the fact that the already known two loop anomalous dimensions of $[27-31]$ are correctly reproduced when the n-dependent results are evaluated for $n = 3$ and $n = 4$. All these checks therefore give us confidence that not only are all our expressions correct but also, for example, that the original Feynman rules were correctly generated.

Having established the $\overline{\text{MS}}$ anomalous dimensions it is then relatively straightforward to deduce the anomalous dimensions in the RI$'$ scheme. This is achieved by replacing the renormalization constants which scale the bare internal parameters and that of the overall quark wave function, by those of the RI$'$ scheme and then impose the RI$'$ scheme definition for the operator renormalization, (2.9). As a check on the resulting renormalization constants, the non-simple poles in ϵ also have to satisfy various constraints emanating from the renormalization group equation, similar to those of the $\overline{\text{MS}}$ scheme. We note, for completeness, that these are fulfilled. Therefore, we record the corresponding three loop RI$'$ anomalous dimensions are, in four dimensions,

$$\gamma^{\text{RI}'}_{\bar{\psi}\gamma^{\mu}D^{\nu}D^{\sigma}\psi}(a) = \frac{25}{6} C_F a + \left[(324\alpha^2 + 972\alpha + 17976) C_A - 2035 C_F - 6744 T_F N_f \right] \frac{C_F a^2}{432}$$
$$+ \left[(29160\alpha^4 + 260820\alpha^3 - 69984\zeta(3)\alpha^2 + 1257768\alpha^2 \right.$$
$$\left. - 723168\zeta(3)\alpha + 4103676\alpha - 6443712\zeta(3) + 50460154) C_A^2 \right.$$

$$
\begin{aligned}
&+ \left(3240\alpha^3 - 91260\alpha^2 - 1043460\alpha - 171072\zeta(3) - 8028146\right) C_A C_F \\
&- \left(259200\alpha^2 - 186624\zeta(3)\alpha + 1401408\alpha + 2322432\zeta(3) \right. \\
&\qquad \left. + 35016976\right) C_A T_F N_f \\
&+ \left(269280\alpha + 3691008\zeta(3) - 3568016\right) C_F T_F N_f \\
&+ \left(2851200\zeta(3) - 1222525\right) C_F^2 + 5492800 T_F^2 N_f^2 \Big] \frac{C_F a^3}{77760} \\
&+ O(a^4)
\end{aligned}
\tag{3.7}
$$

and

$$
\begin{aligned}
\gamma^{\mathrm{RI}'}_{\bar{\psi}\sigma^{\mu\nu}D^\sigma D^\rho \psi}(a) = \ &\frac{13}{3} C_F a \ + \ \left[\left(99\alpha^2 + 297\alpha + 4788\right) C_A - 622 C_F - 1776 T_F N_f\right] \frac{C_F a^2}{108} \\
&+ \Big[\left(8910\alpha^4 + 80055\alpha^3 - 23328\zeta(3)\alpha^2 + 387846\alpha^2 \right. \\
&\qquad \left. - 241056\zeta(3)\alpha + 1279197\alpha - 1902528\zeta(3) + 13940156\right) C_A^2 \\
&+ \left(6210\alpha^3 + 3420\alpha^2 - 157170\alpha + 746496\zeta(3) - 2737412\right) C_A C_F \\
&- \left(79200\alpha^2 - 62208\zeta(3)\alpha + 434736\alpha + 580608\zeta(3) \right. \\
&\qquad \left. + 9592064\right) C_A T_F N_f \\
&+ \left(40560\alpha + 850176\zeta(3) - 706112\right) C_F T_F N_f \\
&+ \left(414720\zeta(3) - 348680\right) C_F^2 + 1491200 T_F^2 N_f^2 \Big] \frac{C_F a^3}{19440} \\
&+ O(a^4)
\end{aligned}
\tag{3.8}
$$

for $n = 3$. Further, for the $n = 4$ operators we have

$$
\begin{aligned}
\gamma^{\mathrm{RI}'}_{\bar{\psi}\gamma^\mu D^\nu D^\sigma D^\rho \psi}(a) = \ &\frac{157}{30} C_F a \ + \ \Big[\left(63000\alpha^2 + 189000\alpha + 2939040\right) C_A - 287303 C_F \\
&\qquad\qquad - 1129560 T_F N_f \Big] \frac{C_F a^2}{54000} \\
&+ \Big[\left(28350000\alpha^4 + 264937500\alpha^3 - 87480000\zeta(3)\alpha^2 \right. \\
&\qquad + 1290594375\alpha^2 - 903960000\zeta(3)\alpha + 4337679375\alpha \\
&\qquad \left. - 4769928000\zeta(3) + 44592646550\right) C_A^2 \\
&+ \left(27675000\alpha^3 + 58466250\alpha^2 - 253773750\alpha \right. \\
&\qquad \left. + 624024000\zeta(3) - 8248198970\right) C_A C_F \\
&- \left(252000000\alpha^2 - 233280000\zeta(3)\alpha + 1452285000\alpha \right. \\
&\qquad \left. + 1995840000\zeta(3) + 31068538400\right) C_A T_F N_f \\
&+ \left(65490000\alpha + 2825280000\zeta(3) - 2407455920\right) C_F T_F N_f \\
&+ \left(1864944000\zeta(3) - 714245693\right) C_F^2 \\
&+ 4979028800 T_F^2 N_f^2 \Big] \frac{C_F a^3}{48600000} \ + \ O(a^4)
\end{aligned}
\tag{3.9}
$$

and

$$
\gamma^{\mathrm{RI}'}_{\bar{\psi}\sigma^{\mu\nu}D^\sigma D^\rho D^\lambda \psi}(a) = \frac{16}{3} C_F a + \left[\left(225\alpha^2 + 675\alpha + 11874\right) C_A - 1184 C_F - 4560 T_F N_f\right] \frac{C_F a^2}{216}
$$

$$
\begin{aligned}
+ \; & \big[\left(16200\alpha^4 + 150768\alpha^3 - 46656\zeta(3)\alpha^2 + 731421\alpha^2 \right. \\
& \quad - \; 482112\zeta(3)\alpha + 2435409\alpha - 3214080\zeta(3) + 27763364 \big)\, C_A^2 \\
& \quad + \; \left(8964\alpha^3 - 8244\alpha^2 - 363072\alpha + 518400\zeta(3) - 5050688 \right) C_A C_F \\
& \quad - \; \left(144000\alpha^2 - 124416\zeta(3)\alpha + 818712\alpha + 1327104\zeta(3) \right. \\
& \qquad \left. + \; 19484432 \right) C_A T_F N_f \\
& \quad + \; \left(93696\alpha + 1990656\zeta(3) - 1687424 \right) C_F T_F N_f \\
& \quad + \; \left(870912\zeta(3) - 376160 \right) C_F^2 + 3138560 T_F^2 N_f^2 \big]\, \frac{C_F a^3}{31104} \\
+ \; & O(a^4)
\end{aligned}
\tag{3.10}
$$

in four dimensions. Clearly they all satisfy the trivial check that the one loop term is scheme independent. Though since the RI$'$ scheme is a mass dependent one, the anomalous dimensions will not necessarily be independent of the gauge parameter as is clearly the case above.

Although we have performed the computation in an arbitrary gauge and colour group, for practical purposes it is useful to specify the results for SU(3). Therefore, the $\overline{\text{MS}}$ transversity anomalous dimensions are

$$
\begin{aligned}
\gamma^{\overline{\text{MS}}}_{\bar{\psi}\sigma^{\mu\nu}D^\sigma D^\rho\psi}(a)\Big|^{\text{SU(3)}} = \; & \frac{52}{9}a \; - \; 2\left[678N_f - 9511 \right] \frac{a^2}{243} \\
& - \; \big[10836N_f^2 + 505440\zeta(3)N_f + 828462N_f \\
& \qquad - \; 51840\zeta(3) - 8885081 \big] \frac{a^3}{6561} \; + \; O(a^4)
\end{aligned}
\tag{3.11}
$$

and

$$
\begin{aligned}
\gamma^{\overline{\text{MS}}}_{\bar{\psi}\sigma^{\mu\nu}D^\sigma D^\rho D^\lambda\psi}(a)\Big|^{\text{SU(3)}} = \; & \frac{64}{9}a \; - \; 2\left[831N_f - 11029 \right] \frac{a^2}{243} \\
& - \; \big[264996N_f^2 + 12441600\zeta(3)N_f + 18758202N_f \\
& \qquad - \; 1360800\zeta(3) - 206734549 \big] \frac{a^3}{131220} \; + \; O(a^4)
\end{aligned}
\tag{3.12}
$$

where $T_F = 1/2$, $C_F = 4/3$ and $C_A = 3$ for SU(3). In addition for the RI$'$ scheme we record each of the anomalous dimensions in the Landau gauge since that is the gauge primarily used in matching to lattice results. We have

$$
\begin{aligned}
\gamma^{\text{RI}'}_{\bar{\psi}\gamma^\mu D^\nu D^\sigma\psi}(a)\Big|^{\text{SU(3)}}_{\alpha=0} = \; & \frac{50}{9}a \; - \; \left[2529N_f - 38411 \right] \frac{a^2}{243} \\
& + \; \big[6179400N_f^2 - 4603392\zeta(3)N_f - 247068636N_f \\
& \qquad - \; 241240032\zeta(3) + 1889349409 \big] \frac{a^3}{262440} \; + \; O(a^4)
\end{aligned}
\tag{3.13}
$$

and

$$
\begin{aligned}
\gamma^{\text{RI}'}_{\bar{\psi}\gamma^\mu D^\nu D^\sigma D^\rho\psi}(a)\Big|^{\text{SU(3)}}_{\alpha=0} = \; & \frac{314}{45}a \; - \; \left[423585N_f - 6325537 \right] \frac{a^2}{30375} \\
& + \; \big[5601407400N_f^2 - 4996080000\zeta(3)N_f - 216935001960N_f \\
& \qquad - \; 167030100000\zeta(3) + 1651820638271 \big] \frac{a^3}{164025000} \\
& + \; O(a^4)
\end{aligned}
\tag{3.14}
$$

for the Wilson operators. Whilst

$$\gamma^{\text{RI}'}_{\bar{\psi}\sigma^{\mu\nu}D^\sigma D^\rho\psi}(a)\Big|^{\text{SU}(3)}_{\alpha=0} = \frac{52}{9}a - 4\left[666N_f - 10151\right]\frac{a^2}{243}$$
$$+ \left[838800N_f^2 - 684288\zeta(3)N_f - 33432384N_f\right.$$
$$\left. - 30148848\zeta(3) + 256256731\right]\frac{a^3}{32805} + O(a^4) \quad (3.15)$$

and

$$\gamma^{\text{RI}'}_{\bar{\psi}\sigma^{\mu\nu}D^\sigma D^\rho D^\lambda\psi}(a)\Big|^{\text{SU}(3)}_{\alpha=0} = \frac{64}{9}a - 5\left[684N_f - 10213\right]\frac{a^2}{243}$$
$$+ \left[1765440N_f^2 - 1492992\zeta(3)N_f - 68291094N_f\right.$$
$$\left. - 56935872\zeta(3) + 515247289\right]\frac{a^3}{52488} + O(a^4) \quad (3.16)$$

for the transversity case.

4. Finite parts

In this section we record the three loop $\overline{\text{MS}}$ and RI$'$ expressions for the amplitudes of the various Green's functions we computed to obtain the previous anomalous dimensions. These are essential for lattice matching computations which therefore necessitates their tedious presentation. The specific definitions of the quantities $\Sigma^{(i)}_{\mathcal{O}}(p)$ are, as noted before, given in appendix A. It is worth pointing out that not all the amplitudes have an a independent leading term.

First, for the Wilson operator with $n = 3$, we have

$$\Sigma^{(1)\,\overline{\text{MS}}\,\text{finite}}_{\bar{\psi}\gamma^\mu D^\nu D^\sigma\psi}(p)\Big|_{p^2=\mu^2} = \left(\frac{1}{3} + \frac{2}{3}\alpha\right)C_F a$$
$$+ \left[\left(\frac{367}{30} - \frac{6}{5}\zeta(3)\alpha + \frac{361}{90}\alpha + \frac{7}{9}\alpha^2 - \frac{6}{5}\zeta(3)\right)C_F C_A\right.$$
$$\left. + \left(-\frac{1087}{120} - \frac{37}{18}\alpha + \frac{1}{9}\alpha^2 + \frac{24}{5}\zeta(3)\right)C_F^2 - \frac{25}{9}N_f T_F C_F\right]a^2$$
$$+ \left[\left(-\frac{36151}{243} + \frac{112}{15}\zeta(3)\alpha - \frac{14671}{810}\alpha + \frac{154}{9}\zeta(3)\right)N_f T_F C_F C_A\right.$$
$$+ \left(\frac{504013}{6480} + \frac{22735}{1944}\alpha - \frac{224}{5}\zeta(3)\right)N_f T_F C_F^2 + \frac{4210}{243}N_f^2 T_F^2 C_F$$
$$+ \left(\frac{6480923}{19440} - \frac{3727}{120}\zeta(3)\alpha + 2\zeta(5)\alpha + \frac{759413}{12960}\alpha\right.$$
$$- \frac{47}{15}\zeta(3)\alpha^2 + \frac{1}{6}\zeta(5)\alpha^2 + \frac{49223}{4320}\alpha^2 + \frac{401}{216}\alpha^3$$
$$\left. - \frac{2563}{24}\zeta(3) - \frac{67}{2}\zeta(5)\right)C_F C_A^2$$
$$+ \left(-\frac{2811619}{6480} + \frac{25}{3}\zeta(3)\alpha + 8\zeta(5)\alpha - \frac{51839}{1215}\alpha - \zeta(3)\alpha^2\right.$$

$$- \frac{613}{162}\alpha^2 - \frac{1}{6}\alpha^3 + \frac{979}{15}\zeta(3) + \frac{784}{3}\zeta(5)\bigg) C_F^2 C_A$$

$$+ \bigg(\frac{28855943}{155520} - 4\zeta(3)\alpha + \frac{218971}{15552}\alpha + \frac{539}{162}\alpha^2$$

$$- \frac{11}{54}\alpha^3 + \frac{860}{9}\zeta(3) - 272\zeta(5) \bigg) C_F^3 \bigg] a^3 \; + \; O(a^4) \quad (4.1)$$

and

$$\Sigma^{(2)\overline{\text{MS}} \text{ finite}}_{\bar{\psi}\gamma^\mu D^\nu D^\sigma \psi}(p)\bigg|_{p^2 = \mu^2} = -\frac{1}{3} + \bigg(\frac{107}{54} + \frac{1}{6}\alpha \bigg) C_F a$$

$$+ \bigg[\bigg(\frac{86597}{4860} + \frac{2}{5}\zeta(3)\alpha + \frac{167}{360}\alpha + \frac{13}{72}\alpha^2 - \frac{18}{5}\zeta(3) \bigg) C_F C_A$$

$$+ \bigg(-\frac{1471891}{155520} - \frac{401}{216}\alpha + \frac{5}{36}\alpha^2 + \frac{12}{5}\zeta(3) \bigg) C_F^2$$

$$- \frac{32363}{3888} N_f T_F C_F \bigg] a^2$$

$$+ \bigg[\bigg(-\frac{30365437}{209952} + \frac{68}{45}\zeta(3)\alpha - \frac{1474}{405}\alpha$$

$$- \frac{3577}{243}\zeta(3) - \frac{100}{9}\zeta(4) \bigg) N_f T_F C_F C_A$$

$$+ \bigg(\frac{1019471}{2099520} - \frac{100}{27}\zeta(3)\alpha + \frac{26413}{2592}\alpha$$

$$+ \frac{4166}{135}\zeta(3) + \frac{100}{9}\zeta(4) \bigg) N_f T_F C_F^2$$

$$+ \bigg(\frac{1227463}{52488} + \frac{400}{243}\zeta(3) \bigg) N_f^2 T_F^2 C_F$$

$$+ \bigg(\frac{208545851}{1049760} - \frac{3721}{720}\zeta(3)\alpha - \frac{1}{8}\zeta(4)\alpha + \frac{1}{6}\zeta(5)\alpha$$

$$+ \frac{390323}{25920}\alpha - \frac{23}{720}\zeta(3)\alpha^2 - \frac{1}{16}\zeta(4)\alpha^2$$

$$- \frac{17}{36}\zeta(5)\alpha^2 + \frac{30983}{8640}\alpha^2 + \frac{1}{9}\zeta(3)\alpha^3 + \frac{475}{864}\alpha^3$$

$$- \frac{31163}{972}\zeta(3) + \frac{647}{144}\zeta(4) + \frac{13}{4}\zeta(5) \bigg) C_F C_A^2$$

$$+ \bigg(-\frac{781692217}{8398080} + \frac{697}{54}\zeta(3)\alpha - \frac{8}{3}\zeta(5)\alpha - \frac{57691}{1620}\alpha$$

$$+ \frac{7}{18}\zeta(3)\alpha^2 - \frac{7895}{2592}\alpha^2 - \frac{1}{3}\zeta(3)\alpha^3 + \frac{1}{24}\alpha^3 - \frac{743}{540}\zeta(3)$$

$$- \frac{67}{6}\zeta(4) - \frac{4}{9}\zeta(5) \bigg) C_F^2 C_A$$

$$+ \bigg(-\frac{1161367}{43740} - \frac{733}{54}\zeta(3)\alpha + \frac{458737}{20736}\alpha - \frac{25}{9}\zeta(3)\alpha^2$$

$$+ \frac{617}{324}\alpha^2 + \frac{2}{9}\zeta(3)\alpha^3 - \frac{31}{216}\alpha^3 + \frac{9203}{972}\zeta(3)$$

$$+ \frac{55}{9}\zeta(4) + 24\zeta(5)\bigg) C_F^3 \bigg] a^3 \; + \; O(a^4) \,. \tag{4.2}$$

For the $n = 4$ case we have

$$\Sigma^{(1)\,\overline{\text{MS}}\text{ finite}}_{\bar{\psi}\gamma^\mu D^\nu D^\sigma D^\rho \psi}(p)\bigg|_{p^2=\mu^2} = -\,1 + \left(\frac{1871}{225} + \frac{4}{3}\alpha\right) C_F a$$

$$+ \bigg[\bigg(\frac{26869109}{324000} - \frac{3}{5}\zeta(3)\alpha + \frac{10313}{1440}\alpha + \frac{245}{144}\alpha^2 - 13\zeta(3)\bigg) C_F C_A$$

$$+ \bigg(-\frac{345682991}{6480000} - \frac{43553}{3600}\alpha + \frac{13}{72}\alpha^2 + \frac{48}{5}\zeta(3)\bigg) C_F^2$$

$$- \frac{6041063}{162000} N_f T_F C_F \bigg] a^2$$

$$+ \bigg[\bigg(-\frac{32796795659}{43740000} + \frac{46}{3}\zeta(3)\alpha - \frac{48917}{1296}\alpha$$

$$- \frac{139334}{2025}\zeta(3) - \frac{628}{15}\zeta(4)\bigg) N_f T_F C_F C_A$$

$$+ \bigg(\frac{63233459093}{437400000} - \frac{628}{45}\zeta(3)\alpha + \frac{2010352}{30375}\alpha$$

$$+ \frac{77018}{675}\zeta(3) + \frac{628}{15}\zeta(4)\bigg) N_f T_F C_F^2$$

$$+ \bigg(\frac{1335574847}{10935000} + \frac{2512}{405}\zeta(3)\bigg) N_f^2 T_F^2 C_F$$

$$+ \bigg(\frac{1578785326351}{1399680000} - \frac{43861}{720}\zeta(3)\alpha - \frac{3}{8}\zeta(4)\alpha + \frac{5}{2}\zeta(5)\alpha$$

$$+ \frac{56000717}{414720}\alpha - \frac{1753}{360}\zeta(3)\alpha^2 - \frac{3}{16}\zeta(4)\alpha^2$$

$$- \frac{4}{3}\zeta(5)\alpha^2 + \frac{986237}{34560}\alpha^2 + \frac{1}{3}\zeta(3)\alpha^3 + \frac{7859}{1728}\alpha^3$$

$$- \frac{2350679}{10125}\zeta(3) + \frac{16687}{1200}\zeta(4) + \frac{91}{3}\zeta(5)\bigg) C_F C_A^2$$

$$+ \bigg(-\frac{1552257600373}{1749600000} + \frac{135041}{2250}\zeta(3)\alpha + 4\zeta(5)\alpha$$

$$- \frac{181148459}{777600}\alpha + \frac{32}{15}\zeta(3)\alpha^2 - \frac{1336787}{51840}\alpha^2$$

$$- \zeta(3)\alpha^3 - \frac{205}{192}\alpha^3 + \frac{395693}{2700}\zeta(3)$$

$$- \frac{1739}{50}\zeta(4) + \frac{116}{3}\zeta(5)\bigg) C_F^2 C_A$$

$$+ \bigg(\frac{203923883969}{2916000000} - \frac{29329}{450}\zeta(3)\alpha + \frac{685201111}{4860000}\alpha$$

$$- \frac{157}{15}\zeta(3)\alpha^2 + \frac{911003}{64800}\alpha^2 + \frac{2}{3}\zeta(3)\alpha^3 - \frac{565}{864}\alpha^3$$

$$- \frac{910609}{40500}\zeta(3) + \frac{1439}{75}\zeta(4) + 96\zeta(5)\bigg) C_F^3 \bigg] a^3$$

$$+O(a^4) \tag{4.3}$$

and

$$\Sigma_{\bar\psi\gamma^\mu D^\nu D^\sigma D^\rho\psi}^{(2)\ \overline{\mathrm{MS}}\ \mathrm{finite}}(p)\Big|_{p^2=\mu^2} = \left(\frac{3}{160} + \frac{1}{32}\alpha\right)C_F a$$

$$+ \left[\left(\frac{70559}{115200} - \frac{3}{40}\zeta(3)\alpha + \frac{4991}{23040}\alpha + \frac{31}{768}\alpha^2 - \frac{1}{40}\zeta(3)\right)C_F C_A\right.$$

$$+ \left(-\frac{40907}{96000} - \frac{7453}{57600}\alpha - \frac{1}{384}\alpha^2 + \frac{1}{5}\zeta(3)\right)C_F^2$$

$$\left. - \frac{119}{800}N_f T_F C_F\right]a^2$$

$$+ \left[\left(-\frac{6448567}{864000} + \frac{53}{120}\zeta(3)\alpha - \frac{104179}{103680}\alpha + \frac{53}{216}\zeta(3)\right)N_f T_F C_F C_A\right.$$

$$+ \left(\frac{28776619}{8640000} + \frac{4054403}{5184000}\alpha - \frac{77}{54}\zeta(3)\right)N_f T_F C_F^2$$

$$+ \left(\frac{818118521}{55296000} - \frac{4003}{2160}\zeta(3)\alpha + \frac{5}{48}\zeta(5)\alpha + \frac{22103567}{6635520}\alpha\right.$$

$$- \frac{6641}{34560}\zeta(3)\alpha^2 + \frac{1}{96}\zeta(5)\alpha^2 + \frac{69635}{110592}\alpha^2 + \frac{913}{9216}\alpha^3$$

$$\left. - \frac{198233}{28800}\zeta(3) + \frac{263}{96}\zeta(5)\right)C_F C_A^2$$

$$+ \left(-\frac{318424073}{25920000} + \frac{7129}{12000}\zeta(3)\alpha + \frac{1}{2}\zeta(5)\alpha - \frac{20896549}{6912000}\alpha\right.$$

$$- \frac{1}{80}\zeta(3)\alpha^2 - \frac{502243}{1382400}\alpha^2 - \frac{29}{1024}\alpha^3 + \frac{1396697}{108000}\zeta(3)$$

$$\left. - 8\zeta(5)\right)C_F^2 C_A + \frac{51959}{54000}N_f^2 T_F^2 C_F$$

$$+ \left(\frac{37693459}{345600000} - \frac{13}{60}\zeta(3)\alpha + \frac{239928131}{207360000}\alpha + \frac{27433}{115200}\alpha^2\right.$$

$$\left.\left. - \frac{41}{4608}\alpha^3 - \frac{272419}{27000}\zeta(3) + \frac{38}{3}\zeta(5)\right)C_F^3\right]a^3$$

$$+ O(a^4)\ . \tag{4.4}$$

Turning to the case of the transversity operator, for $n = 3$ we have

$$\Sigma_{\bar\psi\sigma^{\mu\nu}D^\sigma D^\rho\psi}^{(1)\ \overline{\mathrm{MS}}\ \mathrm{finite}}(p)\Big|_{p^2=\mu^2} = \frac{1}{18}\left[1 + \left[\left(-\frac{109}{18} - \frac{5}{6}\alpha\right)C_F\right]a\right.$$

$$+ \left[\left(-\frac{48941}{810} - \frac{3}{5}\zeta(3)\alpha - \frac{1223}{360}\alpha - \frac{67}{72}\alpha^2 + \frac{59}{5}\zeta(3)\right)C_F C_A\right.$$

$$\left. + \left(\frac{26467}{810} + \frac{119}{18}\alpha - \frac{17}{36}\alpha^2 - \frac{48}{5}\zeta(3)\right)C_F^2 + \frac{2197}{81}N_f T_F C_F\right]a^2$$

$$+ \left[\left(\frac{2254181}{4374} - \frac{124}{15}\zeta(3)\alpha + \frac{32359}{1620}\alpha\right.\right.$$

$$\left. + \frac{3404}{81}\zeta(3) + \frac{104}{3}\zeta(4)\right)N_f T_F C_F C_A$$

$$
\begin{aligned}
+ \ & \left(- \frac{699751}{21870} + \frac{104}{9}\zeta(3)\alpha - \frac{18349}{486}\alpha \right. \\
& \left. - \frac{11176}{135}\zeta(3) - \frac{104}{3}\zeta(4) \right) N_f T_F C_F^2 \\
+ \ & \left(- \frac{177970}{2187} - \frac{416}{81}\zeta(3) \right) N_f^2 T_F^2 C_F \\
+ \ & \left(- \frac{260680459}{349920} + \frac{5537}{180}\zeta(3)\alpha + \frac{3}{8}\zeta(4)\alpha - \frac{3}{2}\zeta(5)\alpha \right. \\
& - \frac{95953}{1296}\alpha + \frac{133}{80}\zeta(3)\alpha^2 + \frac{3}{16}\zeta(4)\alpha^2 + \frac{4}{3}\zeta(5)\alpha^2 \\
& - \frac{35543}{2160}\alpha^2 - \frac{1}{3}\zeta(3)\alpha^3 - \frac{2227}{864}\alpha^3 \\
& \left. + \frac{194707}{1296}\zeta(3) - \frac{463}{48}\zeta(4) - \frac{139}{6}\zeta(5) \right) C_F C_A^2 \\
+ \ & \left(\frac{77900401}{174960} - \frac{3983}{90}\zeta(3)\alpha + 4\zeta(5)\alpha + \frac{635981}{4860}\alpha \right. \\
& - \frac{5}{6}\zeta(3)\alpha^2 + \frac{14591}{1296}\alpha^2 + \zeta(3)\alpha^3 - \frac{1}{24}\alpha^3 - \frac{9163}{135}\zeta(3) \\
& \left. + 22\zeta(4) - 20\zeta(5)) C_F^2 C_A \right. \\
+ \ & \left(\frac{265849}{7290} + \frac{410}{9}\zeta(3)\alpha - \frac{74771}{972}\alpha + \frac{26}{3}\zeta(3)\alpha^2 \right. \\
& - \frac{5063}{648}\alpha^2 - \frac{2}{3}\zeta(3)\alpha^3 + \frac{115}{216}\alpha^3 - \frac{15518}{405}\zeta(3) \\
& \left. \left. - \frac{32}{3}\zeta(4) - 32\zeta(5) \right) C_F^3 \right] a^3 \right] \ + \ O(a^4)
\end{aligned}
\tag{4.5}
$$

with further calculation giving

$$
\begin{aligned}
\Sigma^{(2) \ \overline{\mathrm{MS}} \ \mathrm{finite}}_{\bar\psi\sigma^{\mu\nu}D^\sigma D^\rho \psi}(p) \bigg|_{p^2 = \mu^2} &= \frac{1}{2}\, \Sigma^{(1) \ \overline{\mathrm{MS}} \ \mathrm{finite}}_{\bar\psi\sigma^{\mu\nu}D^\sigma D^\rho \psi}(p) \bigg|_{p^2 = \mu^2} \ + \ O(a^4) \\
\Sigma^{(3) \ \overline{\mathrm{MS}} \ \mathrm{finite}}_{\bar\psi\sigma^{\mu\nu}D^\sigma D^\rho \psi}(p) \bigg|_{p^2 = \mu^2} &= - \frac{3}{2}\, \Sigma^{(1) \ \overline{\mathrm{MS}} \ \mathrm{finite}}_{\bar\psi\sigma^{\mu\nu}D^\sigma D^\rho \psi}(p) \bigg|_{p^2 = \mu^2} \ + \ O(a^4) \, .
\end{aligned}
\tag{4.6}
$$

For the fourth moment of the transversity operator we have

$$
\begin{aligned}
\Sigma^{(1) \ \overline{\mathrm{MS}} \ \mathrm{finite}}_{\bar\psi\sigma^{\mu\nu}D^\sigma D^\rho D^\lambda \psi}(p) \bigg|_{p^2 = \mu^2} = & - \frac{1}{32} \ + \ \left(\frac{293}{1152} + \frac{13}{384}\alpha \right) C_F a \\
& + \left[\left(\frac{2037217}{829440} + \frac{3127}{18432}\alpha + \frac{397}{9216}\alpha^2 - \frac{13}{32}\zeta(3) \right) C_F C_A \right. \\
& \quad + \left(- \frac{8099}{5184} - \frac{1595}{4608}\alpha + \frac{29}{4608}\alpha^2 + \frac{1}{4}\zeta(3) \right) C_F^2 \\
& \quad \left. - \frac{118621}{103680} N_f T_F C_F \right] a^2 \\
& + \left[\left(- \frac{2440299949}{111974400} + \frac{59}{160}\zeta(3)\alpha - \frac{384991}{414720}\alpha \right. \right.
\end{aligned}
$$

$$- \frac{14681}{6480} \zeta(3) - \frac{4}{3} \zeta(4) \Big) N_f T_F C_F C_A$$

$$+ \left(\frac{42119449}{11197440} - \frac{4}{9} \zeta(3)\alpha + \frac{2369273}{1244160} \alpha \right.$$

$$\left. + \frac{4291}{1080} \zeta(3) + \frac{4}{3} \zeta(4) \right) N_f T_F C_F^2$$

$$+ \left(\frac{25433893}{6998400} + \frac{16}{81} \zeta(3) \right) N_f^2 T_F^2 C_F$$

$$+ \left(\frac{112663077941}{3583180800} - \frac{99239}{69120} \zeta(3)\alpha - \frac{3}{256} \zeta(4)\alpha + \frac{5}{96} \zeta(5)\alpha \right.$$

$$+ \frac{89697707}{26542080} \alpha - \frac{2879}{27648} \zeta(3)\alpha^2 - \frac{3}{512} \zeta(4)\alpha^2$$

$$- \frac{17}{384} \zeta(5)\alpha^2 + \frac{541433}{737280} \alpha^2 + \frac{1}{96} \zeta(3)\alpha^3 + \frac{12979}{110592} \alpha^3$$

$$\left. - \frac{503123}{82944} \zeta(3) + \frac{181}{512} \zeta(4) + \frac{463}{384} \zeta(5) \right) C_F C_A^2$$

$$+ \left(- \frac{2225988241}{89579520} + \frac{10093}{5760} \zeta(3)\alpha - \frac{16376411}{2488320} \alpha \right.$$

$$+ \frac{7}{96} \zeta(3)\alpha^2 - \frac{477515}{663552} \alpha^2 - \frac{1}{32} \zeta(3)\alpha^3 - \frac{323}{12288} \alpha^3$$

$$\left. + \frac{1219}{360} \zeta(3) - \frac{27}{32} \zeta(4) + \frac{3}{4} \zeta(5) \right) C_F^2 C_A$$

$$+ \left(\frac{7904501}{3732480} - \frac{295}{144} \zeta(3)\alpha + \frac{1045465}{248832} \alpha - \frac{1}{3} \zeta(3)\alpha^2 \right.$$

$$+ \frac{64399}{165888} \alpha^2 + \frac{1}{48} \zeta(3)\alpha^3 - \frac{1007}{55296} \alpha^3 + \frac{13739}{12960} \zeta(3)$$

$$\left. + \frac{7}{16} \zeta(4) + \frac{5}{6} \zeta(5) \right) C_F^3 \Bigg] a^3 \; + \; O(a^4) \,. \tag{4.7}$$

The remaining amplitudes are related to the first through

$$\Sigma^{(2) \, \overline{\mathrm{MS}} \text{ finite}}_{\bar{\psi}\sigma^{\mu\nu} D^\sigma D^\rho D^\lambda \psi}(p) \bigg|_{p^2 = \mu^2} = - \frac{4}{5} \, \Sigma^{(1) \, \overline{\mathrm{MS}} \text{ finite}}_{\bar{\psi}\sigma^{\mu\nu} D^\sigma D^\rho D^\lambda \psi}(p) \bigg|_{p^2 = \mu^2} \; + \; O(a^4)$$

$$\Sigma^{(3) \, \overline{\mathrm{MS}} \text{ finite}}_{\bar{\psi}\sigma^{\mu\nu} D^\sigma D^\rho D^\lambda \psi}(p) \bigg|_{p^2 = \mu^2} = \frac{1}{5} \, \Sigma^{(1) \, \overline{\mathrm{MS}} \text{ finite}}_{\bar{\psi}\sigma^{\mu\nu} D^\sigma D^\rho D^\lambda \psi}(p) \bigg|_{p^2 = \mu^2} \; + \; O(a^4) \tag{4.8}$$

Finally, for practical purposes we provide the general results for the specific case of the Landau gauge when the colour group is SU(3). Therefore, we have

$$\Sigma^{(1) \, \overline{\mathrm{MS}} \text{ finite}}_{\bar{\psi}\gamma^\mu D^\nu D^\sigma \psi}(p) \bigg|_{p^2 = \mu^2}^{\mathrm{SU}(3), \alpha = 0} = \frac{4}{9} a \; + \; \left[- \frac{50}{27} N_f + \frac{4432}{135} + \frac{56}{15} \zeta(3) \right] a^2$$

$$+ \left[\frac{4210}{729} N_f^2 + \left(- \frac{1665047}{7290} - \frac{28}{5} \zeta(3) \right) N_f \right.$$

$$\left. + \frac{279011797}{131220} - \frac{1717789}{2430} \zeta(3) + \frac{9370}{27} \zeta(5) \right] a^3$$

$$+ \; O(a^4) \tag{4.9}$$

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and

$$\Sigma^{(2)\,\overline{\text{MS}}\,\text{finite}}_{\bar\psi\gamma^\mu D^\nu D^\sigma\psi}(p)\bigg|^{\text{SU(3)},\alpha=0}_{p^2=\mu^2} = -\frac{1}{3} + \frac{214}{81}a + \left[-\frac{32363}{5832}N_f + \frac{4763093}{87480} - \frac{152}{15}\zeta(3)\right]a^2$$
$$+ \left[\left(\frac{1227463}{157464} + \frac{400}{729}\zeta(3)\right)N_f^2\right.$$
$$+ \left(-\frac{1364405723}{4723920} - \frac{814}{405}\zeta(3) - \frac{1000}{81}\zeta(4)\right)N_f$$
$$+ \left(\frac{8619089351}{4723920} - \frac{12125507}{32805}\zeta(3)\right.$$
$$\left.\left. + \frac{8599}{972}\zeta(4) + \frac{2525}{27}\zeta(5)\right)\right]a^3 + O(a^4)\,. \quad (4.10)$$

For the RI′ scheme we note that

$$\Sigma^{(1)\,\text{RI}'\,\text{finite}}_{\bar\psi\gamma^\mu D^\nu D^\sigma\psi}(p)\bigg|^{\text{SU(3)},\alpha=0}_{p^2=\mu^2} = \frac{4}{9}a + \left[-\frac{50}{27}N_f + \frac{44168}{1215} + \frac{56}{15}\zeta(3)\right]a^2$$
$$+ \left[\frac{4210}{729}N_f^2 + \left(-\frac{2738978}{10935} - \frac{28}{5}\zeta(3)\right)N_f\right.$$
$$\left. + \frac{326345791}{131220} - \frac{1678717}{2430}\zeta(3) + \frac{9370}{27}\zeta(5)\right]a^3 + O(a^4)$$
$$\Sigma^{(2)\,\text{RI}'\,\text{finite}}_{\bar\psi\gamma^\mu D^\nu D^\sigma\psi}(p)\bigg|^{\text{SU(3)},\alpha=0}_{p^2=\mu^2} = -\frac{1}{3} + O(a^4)\,. \quad (4.11)$$

Further, for the next Wilson operator

$$\Sigma^{(1)\,\overline{\text{MS}}\,\text{finite}}_{\bar\psi\gamma^\mu D^\nu D^\sigma D^\rho\psi}(p)\bigg|^{\text{SU(3)},\alpha=0}_{p^2=\mu^2} = -1 + \frac{7484}{675}a + \left[-\frac{6041063}{243000}N_f + \frac{431713457}{1822500} - \frac{524}{15}\zeta(3)\right]a^2$$
$$+ \left[\left(\frac{1335574847}{32805000} + \frac{2512}{1215}\zeta(3)\right)N_f^2\right.$$
$$+ \left(-\frac{1349388886469}{984150000} - \frac{43972}{1215}\zeta(3) - \frac{1256}{27}\zeta(4)\right)N_f$$
$$+ \frac{706189399771421}{78732000000} - \frac{112503104}{54675}\zeta(3)$$
$$\left. + \frac{43507}{1620}\zeta(4) + \frac{7180}{9}\zeta(5)\right]a^3 + O(a^4) \quad (4.12)$$

and

$$\Sigma^{(2)\,\overline{\text{MS}}\,\text{finite}}_{\bar\psi\gamma^\mu D^\nu D^\sigma D^\rho\psi}(p)\bigg|^{\text{SU(3)},\alpha=0}_{p^2=\mu^2} = \frac{1}{40}a + \left[-\frac{119}{1200}N_f + \frac{731129}{432000} + \frac{23}{90}\zeta(3)\right]a^2$$
$$+ \left[\frac{51959}{162000}N_f^2 + \left(-\frac{232632277}{19440000} - \frac{755}{972}\zeta(3)\right)N_f\right.$$
$$\left. + \frac{1047728166241}{9331200000} - \frac{109467991}{2916000}\zeta(3) + \frac{13111}{648}\zeta(5)\right]a^3$$
$$+ O(a^4)\,. \quad (4.13)$$

As $\Sigma^{(1)\,\mathrm{RI'\,finite}}_{\bar{\psi}\gamma^\mu D^\nu D^\sigma D^\rho\psi}(p)\Big|^{\mathrm{SU(3)},\alpha=0}_{p^2=\mu^2} = (-1)$ by construction, we note

$$\Sigma^{(2)\,\mathrm{RI'\,finite}}_{\bar{\psi}\gamma^\mu D^\nu D^\sigma D^\rho\psi}(p)\Big|^{\mathrm{SU(3)},\alpha=0}_{p^2=\mu^2} = \frac{1}{40}a + \left[-\frac{119}{1200}N_f + \frac{850873}{432000} + \frac{23}{90}\zeta(3) \right]a^2$$
$$+ \left[\frac{51959}{162000}N_f^2 + \left(-\frac{266088707}{19440000} - \frac{755}{972}\zeta(3) \right)N_f \right.$$
$$\left. + \frac{435587120587}{3110400000} - \frac{20750459}{583200}\zeta(3) + \frac{13111}{648}\zeta(5) \right]a^3$$
$$+ O(a^4) . \tag{4.14}$$

For the transversity cases, when $n = 3$ we have

$$\Sigma^{(1)\,\overline{\mathrm{MS}}\,\mathrm{finite}}_{\bar{\psi}\sigma^{\mu\nu}D^\sigma D^\rho\psi}(p)\Big|^{\mathrm{SU(3)},\alpha=0}_{p^2=\mu^2} = \frac{1}{18}\left[1 - \frac{218}{27}a + \left[\frac{4394}{243}N_f - \frac{669202}{3645} + \frac{452}{15}\zeta(3) \right]a^2 \right.$$
$$+ \left[\left(-\frac{177970}{6561} - \frac{416}{243}\zeta(3) \right)N_f^2 \right.$$
$$+ \left(\frac{98639141}{98415} + \frac{12712}{1215}\zeta(3) + \frac{1040}{27}\zeta(4) \right)N_f$$
$$- \frac{1020141085}{157464} + \frac{59050063}{43740}\zeta(3)$$
$$\left. \left. - \frac{7679}{324}\zeta(4) - \frac{12434}{27}\zeta(5) \right]a^3 \right] + O(a^4) \tag{4.15}$$

and

$$\Sigma^{(1)\,\mathrm{RI'\,finite}}_{\bar{\psi}\sigma^{\mu\nu}D^\sigma D^\rho\psi}(p)\Big|^{\mathrm{SU(3)},\alpha=0}_{p^2=\mu^2} = \frac{1}{18} + O(a^4) . \tag{4.16}$$

Finally, for the transversity moment $n = 4$ we have

$$\Sigma^{(1)\overline{\mathrm{MS}}\,\mathrm{finite}}_{\bar{\psi}\sigma^{\mu\nu}D^\sigma D^\rho D^\lambda\psi}(p)\Big|^{\mathrm{SU(3)},\alpha=0}_{p^2=\mu^2} = -\frac{1}{32} + \frac{293}{864}a + \left[-\frac{118621}{155520}N_f + \frac{13151593}{1866240} - \frac{85}{72}\zeta(3) \right]a^2$$
$$+ \left[\left(\frac{25433893}{20995200} + \frac{16}{243}\zeta(3) \right)N_f^2 \right.$$
$$+ \left(-\frac{20277921581}{503884800} - \frac{1943}{1944}\zeta(3) - \frac{40}{27}\zeta(4) \right)N_f$$
$$+ \frac{2013899793847}{8062156800} - \frac{146176079}{2799360}\zeta(3)$$
$$\left. + \frac{2693}{3456}\zeta(4) + \frac{52991}{2592}\zeta(5) \right]a^3 + O(a^4) \tag{4.17}$$

with clearly

$$\Sigma^{(1)\,\mathrm{RI'\,finite}}_{\bar{\psi}\sigma^{\mu\nu}D^\sigma D^\rho D^\lambda\psi}(p)\Big|^{\mathrm{SU(3)},\alpha=0}_{p^2=\mu^2} = -\frac{1}{32} + O(a^4) . \tag{4.18}$$

5. Conversion functions

An additional check on our computations is provided by the conversion functions for each of the operators we have considered. These functions allow one to convert the anomalous

dimension of the operator in one renormalization scheme to that in another scheme and are defined by the ratio of the renormalization constants in both schemes

$$C_{\mathcal{O}}(a, \alpha) \ = \ \frac{Z_{\mathcal{O}}^{\mathrm{RI}'}}{Z_{\mathcal{O}}^{\overline{\mathrm{MS}}}} \ . \tag{5.1}$$

Then, [37],

$$\gamma_{\mathcal{O}}^{\mathrm{RI}'}\left(a_{\mathrm{RI}'}\right) = \gamma_{\mathcal{O}}^{\overline{\mathrm{MS}}}\left(a_{\overline{\mathrm{MS}}}\right) \ - \ \beta\left(a_{\overline{\mathrm{MS}}}\right) \frac{\partial}{\partial a_{\overline{\mathrm{MS}}}} \ln C_{\mathcal{O}}\left(a_{\overline{\mathrm{MS}}}, \alpha_{\overline{\mathrm{MS}}}\right)$$

$$- \ \alpha_{\overline{\mathrm{MS}}} \gamma_{\alpha}^{\overline{\mathrm{MS}}}\left(a_{\overline{\mathrm{MS}}}\right) \frac{\partial}{\partial \alpha_{\overline{\mathrm{MS}}}} \ln C_{\mathcal{O}}\left(a_{\overline{\mathrm{MS}}}, \alpha_{\overline{\mathrm{MS}}}\right) \tag{5.2}$$

where one needs to express the $\overline{\mathrm{MS}}$ variables in terms of the RI$'$ scheme using (2.3) and (2.4) in order to compare with the anomalous dimensions from the explicit computation. We record that the conversion functions for the various operators we are interested in here are

$$\begin{aligned}
C_{\bar{\psi}\gamma^{\mu}D^{\nu}D^{\sigma}\psi}(a, \alpha) = 1 \ + \ &(27\alpha + 107)\frac{C_F a}{18} \\
&+ \big[\left(86400\alpha^2 - 93312\zeta(3)\alpha + 409104\alpha - 715392\zeta(3) + 3302464\right) C_A \\
&\quad + \left(60480\alpha^2 + 327600\alpha + 373248\zeta(3) + 327549\right) C_F \\
&\quad - \ 1475960 T_F N_f\big] \frac{C_F a^2}{51840} \\
&+ \big[\left(11032200\alpha^3 - 13915152\zeta(3)\alpha^2 - 466560\zeta(5)\alpha^2 + 64538856\alpha^2 \right. \\
&\qquad - 141379344\zeta(3)\alpha + 8398080\zeta(5)\alpha + 319887792\alpha \\
&\qquad - 635381280\zeta(3) + 25660800\zeta(4) + 142767360\zeta(5) \\
&\qquad \left. + 2356357048\right) C_A^2 \\
&\quad + \ \left(4607280\alpha^3 + 5785344\zeta(3)\alpha^2 + 32256792\alpha^2 - 13841280\zeta(3)\alpha \right. \\
&\qquad + 33592320\zeta(5)\alpha + 180233856\alpha - 297743040\zeta(3) \\
&\qquad \left. - 76982400\zeta(4) - 59719680\zeta(5) + 1051633031\right) C_A C_F \\
&\quad + \ \left(35085312\zeta(3)\alpha - 97555104\alpha - 75098880\zeta(3) \right. \\
&\qquad \left. - 93312000\zeta(4) - 1625432200\right) C_A T_F N_f \\
&\quad + \left(2177280\alpha^3 - 23328000\zeta(3)\alpha^2 + 27799200\alpha^2 - 73685376\zeta(3)\alpha \right. \\
&\qquad + 90339057\alpha + 319139136\zeta(3) + 51321600\zeta(4) \\
&\qquad \left. + 201553920\zeta(5) - 607345686\right) C_F^2 \\
&\quad - \ \left(31104000\zeta(3)\alpha + 63327960\alpha - 303948288\zeta(3) \right. \\
&\qquad \left. - 93312000\zeta(4) + 922104436\right) C_F T_F N_f \\
&\quad + \ \left(13824000\zeta(3) + 250653280\right) T_F^2 N_f^2\big] \frac{C_F a^3}{2799360} \ + \ O(a^4) \tag{5.3}
\end{aligned}$$

$$C_{\bar{\psi}\gamma^{\mu}D^{\nu}D^{\sigma}D^{\rho}\psi}(a, \alpha) = 1 \ + \ (525\alpha + 1871)\frac{C_F a}{225}$$

$$
\begin{aligned}
+ \Big[& \big(18315000\alpha^2 - 23328000\zeta(3)\alpha + 88528500\alpha - 103680000\zeta(3) \\
& + 603802180 \big)\, C_A \\
& + \big(21330000\alpha^2 + 119182200\alpha + 62208000\zeta(3) + 98349057 \big)\, C_F \\
& - 264322520\, T_F N_f \Big] \frac{C_F a^2}{6480000} \\
+ \Big[& \big(239334750000\alpha^3 - 340977600000\zeta(3)\alpha^2 - 2916000000\zeta(5)\alpha^2 \\
& + 1428857212500\alpha^2 - 3356364600000\zeta(3)\alpha \\
& + 174960000000\zeta(5)\alpha + 7142849746875\alpha \\
& - 12700608624000\zeta(3) + 335689920000\zeta(4) \\
& + 2504844000000\zeta(5) + 48069511158775 \big)\, C_A^2 \\
& + \big(229047750000\alpha^3 - 142300800000\zeta(3)\alpha^2 + 1689792435000\alpha^2 \\
& \quad - 1524733632000\zeta(3)\alpha + 839808000000\zeta(5)\alpha \\
& \quad + 9165889557000\alpha - 1770530400000\zeta(3) \\
& \quad - 1007069760000\zeta(4) + 653184000000\zeta(5) \\
& \quad + 18744964493980 \big)\, C_A C_F \\
& + \big(816480000000\zeta(3)\alpha - 2158137000000\alpha - 1801163520000\zeta(3) \\
& \quad - 1464998400000\zeta(4) - 31372620527200 \big)\, C_A T_F N_f \\
& + \big(145435500000\alpha^3 - 366249600000\zeta(3)\alpha^2 + 1372611420000\alpha^2 \\
& \quad - 1048904640000\zeta(3)\alpha + 3148063990200\alpha \\
& \quad + 4800009888000\zeta(3) + 671379840000\zeta(4) \\
& \quad + 3359232000000\zeta(5) - 8872064364708 \big)\, C_F^2 \\
& - \big(488332800000\zeta(3)\alpha + 2684380392000\alpha - 4552485120000\zeta(3) \\
& \quad - 1464998400000\zeta(4) + 18121905428720 \big)\, C_F T_F N_f \\
& + \big(217036800000\zeta(3) + 4952079510400 \big)\, T_F^2 N_f^2 \Big] \frac{C_F a^3}{34992000000} \\
+ O(a^4) & \tag{5.4}
\end{aligned}
$$

$$
\begin{aligned}
C_{\bar{\psi}\sigma^{\mu\nu} D^\sigma D^\rho \psi}(a,\alpha) = 1\ +\ & (33\alpha + 109)\frac{C_F a}{18} \\
+ \Big[& \big(6600\alpha^2 - 7776\zeta(3)\alpha + 32067\alpha - 47952\zeta(3) + 228974 \big)\, C_A \\
& + \big(6480\alpha^2 + 30900\alpha + 31104\zeta(3) + 10917 \big)\, C_F \\
& - 99220\, T_F N_f \Big] \frac{C_F a^2}{3240} \\
+ \Big[& \big(3407670\alpha^3 - 4575204\zeta(3)\alpha^2 - 58320\zeta(5)\alpha^2 + 20121777\alpha^2 \\
& \quad - 46022256\zeta(3)\alpha + 2799360\zeta(5)\alpha + 100170405\alpha \\
& \quad - 196675020\zeta(3) + 3732480\zeta(4) + 45081360\zeta(5) \\
& \quad + 693358478 \big)\, C_A^2 \\
& + \big(2334420\alpha^3 - 46656\zeta(3)\alpha^2 + 15956352\alpha^2 - 12324960\zeta(3)\alpha
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \ 11197440\zeta(5)\alpha + 86296752\alpha - 34434720\zeta(3) \\
&\quad - \ 11197440\zeta(4) + 214883180\big)\, C_A C_F \\
&+ (11384064\zeta(3)\alpha - 30726648\alpha - 17280000\zeta(3) \\
&\quad - \ 24261120\zeta(4) - 463372640)\, C_A T_F N_f \\
&+ \big(1399680\alpha^3 - 6065280\zeta(3)\alpha^2 + 13024800\alpha^2 - 13965696\zeta(3)\alpha \\
&\quad + \ 26888652\alpha + 108183168\zeta(3) + 7464960\zeta(4) \\
&\quad + \ 22394880\zeta(5) - 153974772\big)\, C_F^2 \\
&- (8087040\zeta(3)\alpha + 27287280\alpha - 69133824\zeta(3) \\
&\quad - \ 24261120\zeta(4) + 231549328)\, C_F T_F N_f \\
&+ (3594240\zeta(3) + 70515200)\, T_F^2 N_f^2 \Big] \frac{C_F a^3}{699840} \ + \ O(a^4)
\end{aligned}
\tag{5.5}
$$

and

$$
\begin{aligned}
C_{\bar\psi\sigma^{\mu\nu}D^\sigma D^\rho D^\lambda \psi}(a,\alpha) = 1 \ &+ \ (75\alpha + 293)\frac{C_F a}{36} \\
&+ \big[(64890\alpha^2 - 77760\zeta(3)\alpha + 309195\alpha - 414720\zeta(3) + 2302897)C_A \\
&\quad + \ \big(63720\alpha^2 + 380940\alpha + 207360\zeta(3) + 404940\big)\, C_F \\
&\quad - \ 1039688 T_F N_f\big]\frac{C_F a^2}{25920} \\
&+ \ \big[\big(677127600\alpha^3 - 918993600\zeta(3)\alpha^2 - 18662400\zeta(5)\alpha^2 \\
&\quad + \ 4008299580\alpha^2 - 9063653760\zeta(3)\alpha + 466560000\zeta(5)\alpha \\
&\quad + \ 19846116045\alpha - 36380232000\zeta(3) + 783820800\zeta(4) \\
&\quad + \ 8939289600\zeta(5) + 140182687541\big)\, C_A^2 \\
&\quad + \ \big(517071600\alpha^3 - 102643200\zeta(3)\alpha^2 + 3840647400\alpha^2 \\
&\quad\quad - \ 3332482560\zeta(3)\alpha + 2239488000\zeta(5)\alpha + 21797212320\alpha \\
&\quad\quad - \ 9369146880\zeta(3) - 2351462400\zeta(4) \\
&\quad\quad + \ 447897600\zeta(5) + 58907275400\big)\, C_A C_F \\
&\quad + \ (2217093120\zeta(3)\alpha - 6005931840\alpha - 6177116160\zeta(3) \\
&\quad\quad - \ 4777574400\zeta(4) - 94522187168)\, C_A T_F N_f \\
&\quad + \ \big(279936000\alpha^3 - 1194393600\zeta(3)\alpha^2 + 3013848000\alpha^2 \\
&\quad\quad - \ 4503859200\zeta(3)\alpha + 8684647200\alpha + 18380113920\zeta(3) \\
&\quad\quad + \ 1567641600\zeta(4) + 2985984000\zeta(5) - 24416900640\big)\, C_F^2 \\
&\quad - \ (1592524800\zeta(3)\alpha + 6750907200\alpha - 16028098560\zeta(3) \\
&\quad\quad - \ 4777574400\zeta(4) + 57917250880)\, C_F T_F N_f \\
&\quad + \ (707788800\zeta(3) + 15192521216)\, T_F^2 N_f^2\big]\frac{C_F a^3}{111974400} \\
&+ \ O(a^4)
\end{aligned}
\tag{5.6}
$$

where the coupling constant and gauge parameter are in the $\overline{\text{MS}}$ scheme. We note that the same RI$'$ anomalous dimensions are determined as previously.

6. Discussion

We have provided the finite parts of various Green's functions required for the renormalization of the $n = 3$ and 4 moments of the non-singlet twist-2 Wilson and transversity operators at three loops in both the $\overline{\text{MS}}$ and RI' schemes. Since these are available at several loop orders, the hope is that they will be central to the extraction of accurate values for the matrix elements which will be measured on the lattice. From another point of view the new $\overline{\text{MS}}$ anomalous dimensions which are now available for the moments up to and including 4 for the transversity operator will provide a useful check on the full n-dependent three loop transversity anomalous dimensions when they are eventually determined. The impressive symbolic manipulation machinery which achieved the arbitrary n anomalous dimensions for the twist-2 flavour non-singlet and singlet Wilson operators, $[1-4]$, can be applied to the transversity case. Whilst this can be achieved in principle, in the interim one could follow a similar direction to the earlier approach of [39, 40] where the anomalous dimensions of the Wilson operators were determined to moment $n = 10$ and later to higher moments, $n \leq 16$ (except $n = 14$), [45, 46]. These explicit moments were then used to construct solid approximations to the full anomalous dimensions which were then shown to be credible in a substantial range of the x variable in, for example, [47]. Given the advance in computer capabilities since [39, 40], it would seem to us that a fixed moment computation for the anomalous dimensions of the higher moments of the transversity operator is certainly viable. Moreover, one would not be constrained by the choice of an arbitrary covariant gauge and the use of the Feynman gauge would therefore reduce computer time.

Acknowledgments

A. Projection tensors

In this appendix we record the decomposition of the various Green's functions we have computed into the various tensor bases. For the Wilson operators there are two independent tensors consistent with the symmetries and tracelessness conditions of the original operator. By contrast the Green's functions involving the transversity operator require three independent tensors. First, for the Wilson operators, for $n = 3$, we define

$$
\begin{aligned}
\langle \psi(-p) \mathcal{O}_W^{\mu\nu\sigma}(0) \bar{\psi}(p) \rangle = {} & \Sigma^{(1)}_{\bar{\psi}\gamma^\mu D^\nu D^\sigma \psi}(p) \left[p^\mu p^\nu p^\sigma \slashed{p} - \frac{p^2}{(d+2)} \eta^{(\mu\nu} p^{\sigma)} \slashed{p} \right] \frac{1}{(p^2)^3} \\
& + \Sigma^{(2)}_{\bar{\psi}\gamma^\mu D^\nu D^\sigma \psi}(p) \left[\gamma^{(\mu} p^\nu p^{\sigma)} - \frac{2}{(d+2)} \eta^{(\mu\nu} p^{\sigma)} \slashed{p} \right. \\
& \qquad\qquad \left. - \frac{p^2}{(d+2)} \eta^{(\mu\nu} \gamma^{\sigma)} \right] \frac{1}{(p^2)^3} .
\end{aligned}
\tag{A.1}
$$

Then

$$
\begin{aligned}
\Sigma^{(1)}_{\bar\psi\gamma^\mu D^\nu D^\sigma \psi}(p) = & \left[\frac{(d+2)(d+4)}{4(d^2-1)} \left[p^\mu p^\nu p^\sigma \slashed{p} - \frac{p^2}{(d+2)} \eta^{(\mu\nu} p^{\sigma)} \slashed{p} \right] \right. \\
& \left. - \frac{(d+2)p^2}{4(d^2-1)} \left[\gamma^{(\mu} p^\nu p^{\sigma)} - \frac{2}{(d+2)} \eta^{(\mu\nu} p^{\sigma)} \slashed{p} - \frac{p^2}{(d+2)} \eta^{(\mu\nu} \gamma^{\sigma)} \right] \right] \\
& \times \frac{1}{(p^2)^3} \langle \psi(-p) \mathcal{O}_W^{\mu\nu\sigma}(0) \bar\psi(p) \rangle
\end{aligned}
\tag{A.2}
$$

and

$$
\begin{aligned}
\Sigma^{(2)}_{\bar\psi\gamma^\mu D^\nu D^\sigma \psi}(p) = & \left[- \frac{(d+2)}{4(d^2-1)} \left[p^\mu p^\nu p^\sigma \slashed{p} - \frac{p^2}{(d+2)} \eta^{(\mu\nu} p^{\sigma)} \slashed{p} \right] \right. \\
& \left. + \frac{(d+2)p^2}{12(d^2-1)} \left[\gamma^{(\mu} p^\nu p^{\sigma)} - \frac{2}{(d+2)} \eta^{(\mu\nu} p^{\sigma)} \slashed{p} - \frac{p^2}{(d+2)} \eta^{(\mu\nu} \gamma^{\sigma)} \right] \right] \\
& \times \frac{1}{(p^2)^3} \langle \psi(-p) \mathcal{O}_W^{\mu\nu\sigma}(0) \bar\psi(p) \rangle \ .
\end{aligned}
\tag{A.3}
$$

In this appendix and the next in order to compactify our expressions, we note that the symmetrization on Lorentz indices using parentheses excludes the standard normalization factor. So, for instance,

$$
X^{(\mu\nu\sigma)} \ = \ X^{\mu\nu\sigma} \ + \ X^{\nu\sigma\mu} \ + \ X^{\sigma\mu\nu} \ + \ X^{\mu\sigma\nu} \ + \ X^{\sigma\nu\mu} \ + \ X^{\nu\mu\sigma} \ .
\tag{A.4}
$$

Further, in our convention when this operation involves a tensor which is itself symmetric, then one only counts the independent object once. So, for example,

$$
\eta^{\mu(\nu} \eta^{\sigma\rho)} \ = \ \eta^{(\mu\nu} \eta^{\sigma\rho)} \ = \ \eta^{\mu\nu} \eta^{\sigma\rho} \ + \ \eta^{\mu\sigma} \eta^{\nu\rho} \ + \ \eta^{\mu\rho} \eta^{\nu\sigma} \ .
\tag{A.5}
$$

For the $n = 4$ moment of the Wilson operator we decompose the Green's function into two independent tensors with

$$
\langle \psi(-p) \mathcal{O}_W^{\mu\nu\sigma\rho}(0) \bar\psi(p) \rangle \ = \ \sum_{i=1}^{2} \bar\Sigma^{(i)}_{\bar\psi\gamma^\mu D^\nu D^\sigma D^\rho \psi}(p) W_i^{\mu\nu\sigma\rho}(p) \ .
\tag{A.6}
$$

Then each individual amplitude is given by the projections

$$
\begin{aligned}
\bar\Sigma^{(1)}_{\bar\psi\gamma^\mu D^\nu D^\sigma D^\rho \psi}(p) = & \left[\frac{1}{4(d+4)(d+2)(d^2-1)} W_1^{\mu\nu\sigma\rho}(p) - \frac{1}{4(d+4)(d^2-1)p^2} W_2^{\mu\nu\sigma\rho}(p) \right] \\
& \times \langle \psi(-p) \mathcal{O}_{W\,\mu\nu\sigma\rho}(0) \bar\psi(p) \rangle \\
\bar\Sigma^{(2)}_{\bar\psi\gamma^\mu D^\nu D^\sigma D^\rho \psi}(p) = & \left[- \frac{1}{4(d+4)(d^2-1)} W_1^{\mu\nu\sigma\rho}(p) + \frac{(d+3)}{4(d+4)(d^2-1)p^2} W_2^{\mu\nu\sigma\rho}(p) \right] \\
& \times \langle \psi(-p) \mathcal{O}_{W\,\mu\nu\sigma\rho}(0) \bar\psi(p) \rangle
\end{aligned}
\tag{A.7}
$$

where the two basis tensors are

$$
W_1^{\mu\nu\sigma\rho}(p) = \eta^{(\mu\nu} \eta^{\sigma\rho)} \slashed{p} \ - \ \frac{(d+2)}{2} \eta^{(\mu\nu} p^\sigma \gamma^{\rho)} \ + \ \frac{(d+2)(d+4)}{2p^2} p^{(\mu} p^\nu p^\sigma \gamma^{\rho)}
$$

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$$- \frac{(d+2)(d+4)}{(p^2)^2} p^\mu p^\nu p^\sigma p^\rho \not{p}$$

$$W_2^{\mu\nu\sigma\rho}(p) = \eta^{(\mu\nu} p^\sigma p^{\rho)} \not{p} - \frac{p^2}{2} \eta^{(\mu\nu} p^\sigma \gamma^{\rho)} + \frac{(d+4)}{2} p^{(\mu} p^\nu p^\sigma \gamma^{\rho)}$$

$$- \frac{2(d+4)}{p^2} p^\mu p^\nu p^\sigma p^\rho \not{p} . \tag{A.8}$$

Finally, for this particular case to ensure that the tree term of one of the amplitudes involves unity, we have chosen to redefine the amplitudes as

$$\bar{\Sigma}^{(1)}_{\bar{\psi}\gamma^\mu D^\nu D^\sigma D^\rho \psi}(p) = - (d+2)(d+4) \left[\Sigma^{(1)}_{\bar{\psi}\gamma^\mu D^\nu D^\sigma D^\rho \psi}(p) - \frac{1}{2(d+4)} \Sigma^{(2)}_{\bar{\psi}\gamma^\mu D^\nu D^\sigma D^\rho \psi}(p) \right]$$

$$\bar{\Sigma}^{(2)}_{\bar{\psi}\gamma^\mu D^\nu D^\sigma D^\rho \psi}(p) = \Sigma^{(2)}_{\bar{\psi}\gamma^\mu D^\nu D^\sigma D^\rho \psi}(p) . \tag{A.9}$$

Turning to the cases for the transversity operators then for $n = 3$ the decomposition is

$$\langle \psi(-p) \mathcal{O}_T^{\mu\nu\sigma\rho}(0) \bar{\psi}(p) \rangle = \sum_{i=1}^{3} \Sigma^{(i)}_{\bar{\psi}\sigma^{\mu\nu} D^\sigma D^\rho \psi}(p) T_i^{(3)\mu\nu\sigma\rho}(p) . \tag{A.10}$$

Hence the coefficients are given by the three projections of the Green's function

$$\Sigma^{(1)}_{\bar{\psi}\sigma^{\mu\nu} D^\sigma D^\rho \psi}(p) = \left[- \frac{1}{12(d^2-4)(d^2-1)} T_1^{(3)\mu\nu\sigma\rho}(p) - \frac{1}{12(d^2-4)(d^2-1)} T_2^{(3)\mu\nu\sigma\rho}(p) \right]$$

$$\times \langle \psi(-p) \mathcal{O}_{T\,\mu\nu\sigma\rho}(0) \bar{\psi}(p) \rangle$$

$$\Sigma^{(2)}_{\bar{\psi}\sigma^{\mu\nu} D^\sigma D^\rho \psi}(p) = \left[- \frac{1}{12(d^2-4)(d^2-1)} T_1^{(3)\mu\nu\sigma\rho}(p) \right.$$

$$- \frac{(d^3+d^2-2d+4)}{12d(d+4)(d^2-4)(d^2-1)} T_2^{(3)\mu\nu\sigma\rho}(p)$$

$$\left. - \frac{1}{12d(d+4)(d-1)} T_3^{(3)\mu\nu\sigma\rho}(p) \right] \langle \psi(-p) \mathcal{O}_{T\,\mu\nu\sigma\rho}(0) \bar{\psi}(p) \rangle$$

$$\Sigma^{(3)}_{\bar{\psi}\sigma^{\mu\nu} D^\sigma D^\rho \psi}(p) = \left[- \frac{1}{12d(d+4)(d-1)} T_2^{(3)\mu\nu\sigma\rho}(p) - \frac{(d^2+2)}{12d(d+4)(d-2)(d^2-1)} T_3^{(3)\mu\nu\sigma\rho}(p) \right]$$

$$\times \langle \psi(-p) \mathcal{O}_{T\,\mu\nu\sigma\rho}(0) \bar{\psi}(p) \rangle \tag{A.11}$$

where the choice of basis tensors is

$$T_1^{(3)\mu\nu\sigma\rho}(p) = - \frac{(d+2)(d+4)}{p^2} \sigma^{\lambda\mu} p^\nu p^\sigma p^\rho p_\lambda + (d+2) \sigma^{\lambda\mu} \eta^{(\nu\sigma} p^{\rho)} p_\lambda$$

$$- (d+2) \sigma^{\mu(\nu} p^\sigma p^{\rho)} + \sigma^{\mu(\nu} \eta^{\sigma\rho)} p^2$$

$$T_2^{(3)\mu\nu\sigma\rho}(p) = \frac{(d+1)(d+2)}{p^2} \sigma^{\lambda\mu} p^\nu p^\sigma p^\rho p_\lambda - (d+1) \sigma^{\lambda\mu} \eta^{(\nu\sigma} p^{\rho)} p_\lambda$$

$$+ \frac{(d+2)}{p^2} \sigma^{\lambda(\nu} p^\sigma p^{\rho)} p^\mu p_\lambda - \sigma^{\mu(\nu} \eta^{\sigma\rho)} p^2$$

$$T_3^{(3)\mu\nu\sigma\rho}(p) = - \frac{(d^2+2d-2)}{p^2} \sigma^{\lambda\mu} p^\nu p^\sigma p^\rho p_\lambda + d \sigma^{\lambda\mu} \eta^{(\nu\sigma} p^{\rho)} p_\lambda$$

$$+ \frac{2}{p^2} \sigma^{\lambda(\nu} p^\sigma p^{\rho)} p^\mu p_\lambda - \sigma^{\lambda(\nu} \eta^{|\mu|\sigma} p^{\rho)} p_\lambda . \tag{A.12}$$

Finally, for the $n = 4$ moment of the transversity operator we set

$$\langle \psi(-p) \mathcal{O}_T^{\mu\nu\sigma\rho\lambda}(0) \bar{\psi}(p) \rangle \;=\; \sum_{i=1}^{3} \Sigma^{(i)}_{\bar{\psi}\sigma^{\mu\nu}D^\sigma D^\rho D^\lambda \psi}(p) T_i^{\mu\nu\sigma\rho\lambda}(p) \tag{A.13}$$

where the three amplitudes are given by

$$\Sigma^{(1)}_{\bar{\psi}\sigma^{\mu\nu}D^\sigma D^\rho D^\lambda \psi}(p) = -\;\frac{1}{16(d+4)(d^2-1)(d-2)} T_1^{\mu\nu\sigma\rho\lambda}(p)\langle \psi(-p) \mathcal{O}_{T\,\mu\nu\sigma\rho\lambda}(0)\bar{\psi}(p)\rangle$$

$$\Sigma^{(2)}_{\bar{\psi}\sigma^{\mu\nu}D^\sigma D^\rho D^\lambda \psi}(p) = \left[-\;\frac{(d^2 + 7d + 14)}{4(d+6)(d+4)(d+3)(d+1)(d^2-1)(d-2)} T_2^{\mu\nu\sigma\rho\lambda}(p) \right.$$
$$\left. +\;\frac{(d+2)}{2(d+6)(d+4)(d+3)(d+1)(d^2-1)(d-2)} T_3^{\mu\nu\sigma\rho\lambda}(p) \right]$$
$$\times\; \langle \psi(-p)\mathcal{O}_{T\,\mu\nu\sigma\rho\lambda}(0)\bar{\psi}(p)\rangle$$

$$\Sigma^{(3)}_{\bar{\psi}\sigma^{\mu\nu}D^\sigma D^\rho D^\lambda \psi}(p) = \left[\frac{(d+2)}{2(d+6)(d+4)(d+3)(d+1)(d^2-1)(d-2)} T_2^{\mu\nu\sigma\rho\lambda}(p) \right.$$
$$\left. -\;\frac{(d^2 + d + 2)}{16(d+6)(d+4)(d+3)(d+1)(d^2-1)(d-2)} T_3^{\mu\nu\sigma\rho\lambda}(p) \right]$$
$$\times\; \langle \psi(-p)\mathcal{O}_{T\,\mu\nu\sigma\rho\lambda}(0)\bar{\psi}(p)\rangle \tag{A.14}$$

in terms of the three basis tensors

$$T_1^{\mu\nu\sigma\rho\lambda}(p) = -\;\frac{4(d+4)}{p^2}\sigma^{\theta\mu}p^\nu p^\sigma p^\rho p^\lambda p_\theta \;+\; 2\sigma^{\theta\mu}\eta^{(\nu\sigma}p^\rho p^{\lambda)}p_\theta \;+\; \frac{(d+4)}{p^2}\sigma^{\theta(\nu}p^\sigma p^\rho p^{\lambda)}p^\mu p_\theta$$
$$-\sigma^{\theta(\nu}\eta^{\sigma\rho}p^{\lambda)}p^\mu p_\theta \;-\; (d+4)\sigma^{\mu(\nu}p^\sigma p^\rho p^{\lambda)} \;+\; p^2\sigma^{\mu(\nu}\eta^{\sigma\rho}p^{\lambda)}$$

$$T_2^{\mu\nu\sigma\rho\lambda}(p) = -\;\frac{(d+2)(d+4)}{p^2}\sigma^{\theta\mu}p^\nu p^\sigma p^\rho p^\lambda p_\theta \;-\; p^2\sigma^{\theta\mu}\eta^{(\nu\sigma}\eta^{\rho\lambda)}p_\theta \;+\; (d+2)\sigma^{\theta\mu}\eta^{(\nu\sigma}p^\rho p^{\lambda)}p_\theta$$
$$-\frac{(d+4)}{p^2}\sigma^{\theta(\nu}p^\sigma p^\rho p^{\lambda)}p^\mu p_\theta \;+\; \sigma^{\theta(\nu}\eta^{\sigma\rho}p^{\lambda)}p^\mu p_\theta$$

$$T_3^{\mu\nu\sigma\rho\lambda}(p) = -\;\frac{4(d+4)}{p^2}\sigma^{\theta\mu}p^\nu p^\sigma p^\rho p^\lambda p_\theta \;+\; 2\sigma^{\theta\mu}\eta^{(\nu\sigma}p^\rho p^{\lambda)}p_\theta \;-\; p^2\sigma^{\theta(\nu}\eta^{|\mu|\sigma}\eta^{\rho\lambda)}p_\theta$$
$$-\frac{(d+4)(d+5)}{p^2}\sigma^{\theta(\nu}p^\sigma p^\rho p^{\lambda)}p^\mu p_\theta \;+\; (d+4)\sigma^{\theta(\nu}\eta^{|\mu|\sigma}p^\rho p^{\lambda)}p_\theta \;. \tag{A.15}$$

Finally, it is worth observing that in the two loop computation of the anomalous dimensions of [5–7, 27–31] the Feynman rules were constructed by introducing a null vector Δ_μ which projected out a part of the Green's function which had a non-zero tree term and therefore allowed for the extraction of the anomalous dimension. Since the lattice matching specifically requires information on the full Lorentz structure of the Green's function we cannot follow that approach as the null vector would exclude access to several of the amplitudes which are extracted from the data by choosing different momentum configurations.

B. Operators

In this section we list the full form of the four operators we considered which satisfy the symmetrization and tracelessness properties. For the Wilson operators we have

$$\mathcal{O}_W^{\mu\nu\sigma} \;=\; W^{\mu\nu\sigma} \;-\; \frac{1}{(d+2)}\eta^{(\mu\nu}W^{\sigma)\theta}{}_\theta \tag{B.1}$$

for $n = 3$ where

$$W^{\mu\nu\sigma} = \frac{1}{6}\bar{\psi}\gamma^{(\mu}D^{\nu}D^{\sigma)}\psi \tag{B.2}$$

and for $n = 4$

$$\mathcal{O}_W^{\mu\nu\sigma\rho} = W^{\mu\nu\sigma\rho} - \frac{1}{(d+4)}\eta^{(\mu\nu}W^{\sigma\rho)\theta}{}_\theta + \frac{1}{(d+2)(d+4)}\eta^{(\mu\nu}\eta^{\sigma\rho)}W^\theta{}_\theta{}^\phi{}_\phi \tag{B.3}$$

where

$$W^{\mu\nu\sigma\rho} = \frac{1}{24}\bar{\psi}\gamma^{(\mu}D^{\nu}D^{\rho}D^{\sigma)}\psi \ . \tag{B.4}$$

For the transversity operators, which have one fewer traceless conditions compared to the Wilson operators, the full operators are more involved. For $n = 3$ we have

$$\begin{aligned}
\mathcal{O}_T^{\mu\nu\sigma\rho} = T^{\mu\nu\sigma\rho} &- \frac{(d^2+4d+2)}{d(d+2)(d+4)}T^{\mu(\nu|\theta}{}_{\theta|}\eta^{\sigma\rho)} - \frac{2}{d(d+2)(d+4)}T^{(\nu|\mu\theta}{}_{\theta|}\eta^{\sigma\rho)}\\
&+ \frac{1}{d(d+4)}T^{(\nu\sigma|\theta}{}_\theta\eta^{\mu|\rho)} + \frac{2}{d(d+4)}T^{\theta\mu(\nu}{}_{|\theta|}\eta^{\sigma\rho)}\\
&- \frac{(d+2)}{d(d+4)}T^{\theta(\nu\sigma}{}_{|\theta}\eta^{\mu|\rho)} + \frac{1}{(d+2)(d+4)}T^{\theta\phi}{}_{\theta\phi}\eta^{\mu(\nu}\eta^{\sigma\rho)}
\end{aligned} \tag{B.5}$$

where

$$T^{\mu\nu\sigma\rho} = \frac{1}{6}\bar{\psi}\sigma^{\mu(\nu}D^\sigma D^{\rho)}\psi \ . \tag{B.6}$$

Finally, for $n = 4$ the fully symmetrized and traceless operator is

$$\begin{aligned}
\mathcal{O}_T^{\mu\nu\sigma\rho\lambda} = T^{\mu\nu\sigma\rho\lambda} &- \frac{(d^2+7d+8)}{(d+1)(d+4)(d+6)}T^{\mu(\nu\sigma|\theta}{}_{\theta|}\eta^{\rho\lambda)}\\
&+ \frac{(d+5)}{(d+1)(d+4)(d+6)}T^{\mu\theta}{}_\theta{}^\phi{}_\phi\eta^{(\nu\sigma}\eta^{\rho\lambda)} - \frac{2}{(d+1)(d+4)(d+6)}T^{(\nu|\mu|\sigma|\theta}{}_{\theta|}\eta^{\rho\lambda)}\\
&+ \frac{1}{(d+1)(d+6)}T^{(\nu\sigma\rho|\theta}{}_\theta\eta^{\mu|\lambda)} - \frac{1}{(d+1)(d+4)(d+6)}T^{(\nu|\theta}{}_\theta{}^\phi{}_\phi\eta^{\mu|\sigma}\eta^{\rho\lambda)}\\
&+ \frac{2}{(d+1)(d+6)}T^{\theta\mu(\nu\sigma}{}_{|\theta|}\eta^{\rho\lambda)} - \frac{4}{(d+1)(d+4)(d+6)}T^{\theta\mu}{}_\theta{}^\phi{}_\phi\eta^{(\nu\sigma}\eta^{\rho\lambda)}\\
&- \frac{(d+4)}{(d+1)(d+6)}T^{\theta(\nu\sigma\rho}{}_{|\theta}\eta^{\mu|\lambda)} + \frac{(d+2)}{(d+1)(d+4)(d+6)}T^{\theta(\nu|}{}_\theta{}^\phi{}_\phi\eta^{\mu|\sigma}\eta^{\rho\lambda)}
\end{aligned} \tag{B.7}$$

where

$$T^{\mu\nu\sigma\rho\lambda} = \frac{1}{24}\bar{\psi}\sigma^{\mu(\nu}D^\rho D^\sigma D^{\lambda)}\psi \ . \tag{B.8}$$

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